# JIWAJI UNIVERSITY GWALIOR

# SELF LEARNING MATERIAL

# FOR

## B.SC. 1 YEAR PHYSICS

## PAPER 101: Mathematical Physics, Mechanics and Properties of Matter

## PAPER CODE: 101

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# PAPER 101: Mathematical Physics, Mechanics and Properties of Matter

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**WRITER**

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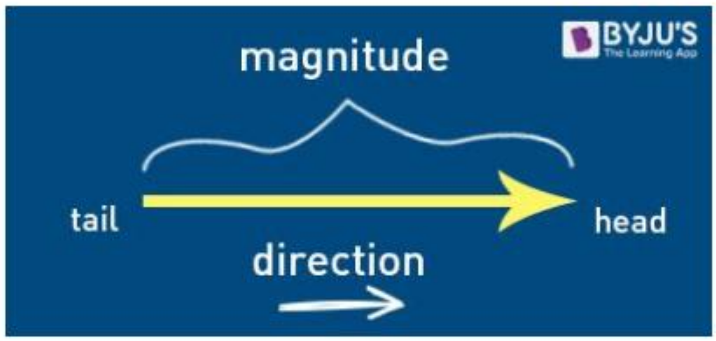
**Master of Computer Application**

# UNIT-1

## Vectors

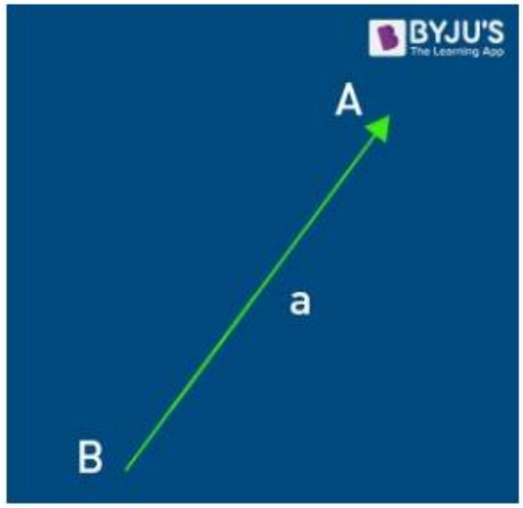
A vector describes a movement from one point to another. It is a mathematical quantity having both Magnitude & Direction. The length of the segment of the directed line is called the magnitude of the vector and the angle at which the vector is inclined shows the direction of the vector.




The beginning point of a vector is called as "Tail" and the end side (having arrow) is called as "Head."

## Examples & Representation-

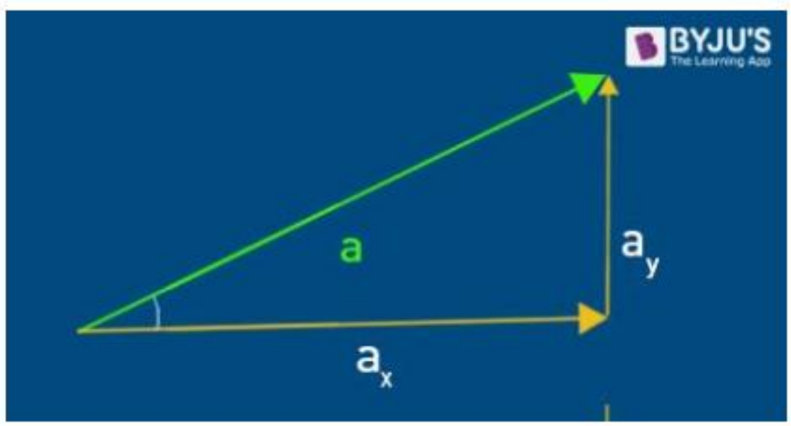
Velocity, Acceleration, Force, Increase/Decrease in Temperature etc.




A vector between two points A and B is given as $\overrightarrow{AB}$, or vector a.

## Understanding more about Vectors-

Breaking a vector into its x and y components is the most common way for solving vectors.




A vector "a" is inclined with horizontal having an angle equal to $\theta$.

This given vector "a" can be broken down into two components i.e. $a_x$ and $a_y$.

The component $a_x$ is called as "Horizontal component" whose value is $a \cos \theta$.

The component $a_y$ is called as "Vertical component" whose value is $a \sin \theta$.

**Example- Given vector V, having magnitude of 10 units & inclined at $60^{\circ}$. Break down the given vector into its two component.**

**Solution-** $\overrightarrow{v}$, having magnitude(V) = 10 units and $\theta = 60^{\circ}$

Horizontal component ($V_x$) = $V \cos \theta$

$V_x$ = $10 \ \cos 60^{\circ}$

$V_x$ = $10 \times 0.5$

$V_x$ = 5 units

Now, Vertical component($V_y$) = $V \sin \theta$

$V_y$ = $10 \ \sin 60^{\circ}$

$V_y$ = $10 \times \frac{\sqrt{3}}{2}$

$V_y$ = $10\sqrt{3}$ units

## Magnitude of a Vector-

The magnitude of a vector is shown by vertical lines on both the sides of the given vector.

$|a|$

Mathematically, the magnitude of a vector is calculated by the help of "Pythagoras Theorem," i.e.

$|a| = \sqrt{x^2 + y^2}$,

**Example- Find the magnitude of vector a (3,4).**

**Solution-**

Given $\overrightarrow{a}$ = (3,4)

$|a| = \sqrt{x^2 + y^2}$

$|a| = \sqrt{3^2 + 4^2}$

$\Rightarrow |a| = \sqrt{9 + 16} = \sqrt{25}$

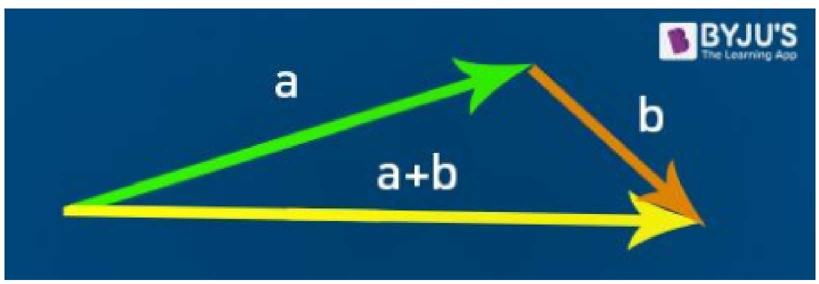
Therefore, $|a| = 5$

## Operation on Vector–

Vector operation such as Addition, Subtraction, Multiplication etc. can be done easily.

### 1. Addition of Vectors-

The two vectors a and b can be added giving the sum to be a + b. This requires joining them head to tail.



We can translate the vector b till its tail meets the head of a. The line segment that is directed from the tail of vector a to the head of vector b is the vector "a + b".

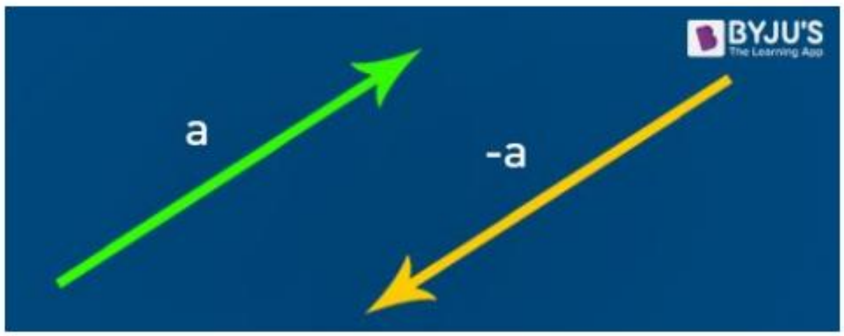
### Characteristics of Vector Addition-

- Commutative Law- the order of addition does not matter, i.e, a + b = b + a
- Associative law- the sum of three vectors has nothing to do with which pair of the vectors is added at the beginning.

i.e. (a + b) + c = a + (b + c)

## 2. Subtraction of Vectors-

Before going to the operation it is necessary to know about reverse vector(-a).



A reverse vector (-a) which is opposite of a has similar magnitude as a but pointed in opposite direction.

First, we find the reverse vector.

Then add them as the usual addition.

Such as if we wanna find vector b – a

Then, b – a = b + (-a)

## 3. Multiplication of Vectors-

- **Scalar Multiplication-**

Multiplication of a vector by a scalar quantity is called "Scaling."

In this type of multiplication, only the magnitude of a vector is changed not the direction.

- **Vector Multiplication-**

It is of two types "Cross product" and "Dot product."

## Cross Product-

The cross product of two vectors results in a vector quantity. It is represented by a cross sign between two vectors.

$$a \times b$$

Mathematical value of a cross product-

$$a \times b = |a|\,|b| \sin\theta\,\hat{n}$$

where, $|a|$ is the magnitude of vector a.

$|a|$ is the magnitude of vector b.

$\theta$ is the angle between two vectors a & b.

and $\hat{n}$ is a unit vector showing the direction of the multiplication of two vectors.

## Dot product-

The dot product of two vectors always result in scalar quantity, i.e. it has only magnitude and no direction. It is represented by a dot in between two vectors.

$$a.b$$

**Mathematical value-**

$$a.b = |a|\,|b| \cos\theta$$

**Example- Find the scalar and vector multiplication of two vectors a and b given by $3i - 1j + 2k$ and $1\hat{i} + -2\hat{j} + 3\hat{k}$ respectively.**

**Solution-**

Given vector a (3,-1,2) and vector b (1,-2,3)

Vector product ( or Cross product) = $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 1 & -2 & 3 \end{vmatrix}$

$\vec{a} \times \vec{b} = \begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} \hat{k}$

$\vec{a} \times \vec{b} = 1\hat{i} - (7)\hat{j} + (-5)\hat{k}$

$\vec{a} \times \vec{b}$ = $1\hat{i} - 7\hat{j} - 5\hat{k}$

Now, $\left|\vec{a} \times \vec{b}\right| = \sqrt{(1)^2 + (-7)^2 + (-5)^2}$

$\left|\vec{a} \times \vec{b}\right|$ = $\sqrt{75} = 5\sqrt{3}$

Now finding magnitudes of vector a and b

$|\vec{a}| = \sqrt{(3)^2 + (-1)^2 + (2)^2}$

$|\vec{a}|$ = $\sqrt{9 + 1 + 4} = \sqrt{14}$

$\left|\vec{b}\right| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$

$\left|\vec{b}\right|$ = $\sqrt{1 + 4 + 9} = \sqrt{14}$

$\sin\theta$ = $\frac{\left|\vec{a} \times \vec{b}\right|}{|\vec{a}| \times \left|\vec{b}\right|}$

$\sin\theta = \frac{5\sqrt{3}}{\sqrt{14} \times \sqrt{14}}$

$\sin\theta = \frac{5\sqrt{3}}{14}$

Or, $\theta$ = $\sin^{-1}\left(\frac{5\sqrt{3}}{14}\right)$

Thus the Vector product is equal to $1\hat{i} - 7\hat{j} – 5\hat{k}$

Scalar product (or Dot product) = $\vec{a}.\vec{b} = |a|\,|b|\cos\theta$

Where $\theta$ is the angle between the vectors. But we don't know the angle between the vectors thus another method of multiplication can be used.

$\vec{a}.\vec{b} = (3\hat{i} - 1\hat{j} + 2\hat{k}).(1\hat{i} - 2\hat{j} + 3\hat{k})$

$\vec{a}.\vec{b} = 3(\hat{i}.\hat{i}) + 2(\hat{j}.\hat{j}) + 6(\hat{k}.\hat{k})$

$\vec{a}.\vec{b} = 3 + 2 + 6$

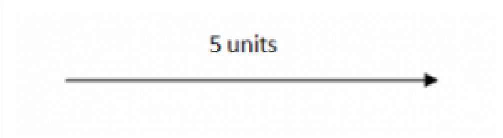
$\vec{a}.\vec{b} = 11$

## Vector

Vector in physics can be defined as a quantity comprised of both direction and magnitude. Apart from direction and magnitude vector not have any position, in simple it is not altered unless and until it is not displaced parallel to itself. A vector is represented by a symbol arrow whose length will be proportional to the quantity's magnitude and lies in the same direction of the quantity.

Some of the vector quantities include force, displacement, acceleration and distance. Scaled vector diagrams are used to represent vector quantities. The magnitude of the vector is usually represented by the length of the arrow in a scalar vector diagram.

Representation of Vectors: Take velocity vector for an instant, if we want to represent a velocity vector of magnitude five units and along the direction of a positive x axis. This can be represented by drawing a line parallel to velocity and putting an arrow showing the direction of velocity.



The vectors are denoted by putting an arrow over the symbols representing them. Thus, we write $\overrightarrow{AB}$, $\overrightarrow{BC}$ etc. Sometimes a vector is represented by a single letter such as:

Force vector: $\vec{F}$

Velocity vector: $\vec{v}$

Acceleration vector: $\vec{a}$

Linear momentum: $\vec{p}$

## Equality of Vectors

Two vectors are said to be equal if their magnitudes and directions are same. Here we are talking about two values of the same physical quantity, i.e. we can not talk about equality of two vectors if they don't represent the same physical quantity. For instance, one can't say that velocity vector of 5 m/s in the positive x-axis and Force vector of 5 N also in positive x-axis are equal.

# Polar and axial vectors

These notions refer to the properties of vectors under *reflections*, their *parity properties*. One can consider reflection of the physical system in a plane [Hylleraas 1950], with the coordinate system fixed, or equivalently inversion of all coordinate axes [Goldstein 1980] with the physical system fixed. We will use the latter in what follows.

Let $\boldsymbol{r}$ denote a position vector in a 3D Euclidean space, and $\boldsymbol{p}$ a momentum vector in the point. After inversion of the coordinate directions the two vectors' directions in space will be unchanged, so the transformation rule will be:

- *Polar* vectors:

$$\begin{aligned} x_i &\rightarrow x_i' = -x_i \\ p_i &\rightarrow p_i' = -p_i \end{aligned} \tag{D.1}$$

The angular momentum $\boldsymbol{L}$,

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{r} \times \boldsymbol{p} \\ L_i &= \epsilon_{ijk} x_j p_k \end{aligned}$$

has a different rule of transformation:

- *Axial* vectors:

$$
\begin{aligned}
L_i \quad \rightarrow \quad L'_i &= \epsilon_{ijk} x'_j p'_k \\
&= \epsilon_{ijk} x_j p_k \\
&= L_i
\end{aligned}
\tag{D.2}
$$

The components of polar vectors change sign during the inversion, those of axial vectors do not. Among examples of polar vectors in these lecture notes:

$$\boldsymbol{r}\,,\ \nabla\,,\ \boldsymbol{u}$$

One example of an axial vector is the vorticity:

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$$

(also a vector product). The vector product of two polar or two axial vectors will be an axial vector; considerations of sign will correspondingly show that the vector product of one polar and one axial vector will be a polar vector.[1]

Axial vectors have an associated direction of rotation, a *handedness*: An axial vector can equivalently be expressed by the components of an antisymmetric tensor. See the example in Appendix B,

$$\eta_{ij} = \frac{1}{2}\epsilon_{ijk}\omega_k$$

where the antisymmetric tensor $\eta_{ij}$ is a measure for the rotational property of an arbitrary velocity field, like $\boldsymbol{\omega}$.

A lefthanded coordinate system results from the inversion of a righthanded one. An axial vector thus changes handedness because its components, referred to the coordinate systems, do not change.

If one side of a vector equation decribing a physical relation is polar (or axial), then also the other side must be polar (or axial). That follows because a physical process and a reflected copy of it must be physically equivalent.[2] In Chapter 4 that requirement is used in the derivation of the velocity field for Stokes flow.

In the preceding text we got a glimpse of some fundamental relations between vectors and the structures called tensors.[3] Both can be related to the notion of *differential forms*, mathematical structures which however are well beyond the scope of these lecture notes. The introduction of tensors in this course is of the traditional operational kind for fluid mechanics: They are structures which arise in a natural way in the description of tensions in a surface, including supporting structures like Levi-Civitá's which contribute to simplifying the notation. On that background it is not easy to give a consise reason for an equation like (B.12), beyond stating that it has a form which does not favorize any direction in space compared to the others. In other derivations of Eq. (B.13) at the same level of description, see for instance [Papatzacos 2003], one neither avoids postulates.

The energetic student is therefore advised to indulge in a course of diffential surface geometry!

## Triple product

### Scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

- Scalar triple product given by the true determinant

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Your knowledge of determinants tells you that if you
  - swap one pair of rows of a determinant, sign changes;
  - swap two pairs of rows, its sign stays the same.
- Hence

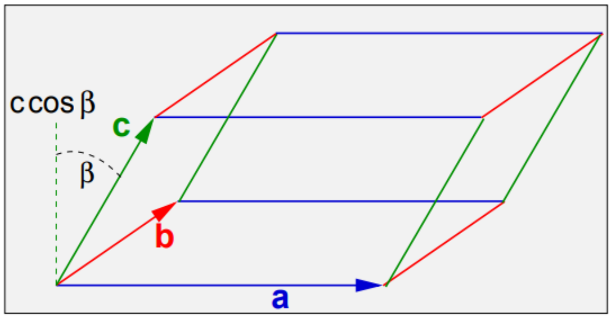
(i) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ (Cyclic permutation.)

(ii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$ and so on. (Anti-cyclic permutation)

(iii) The fact that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ allows the scalar triple product to be written as $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$. This notation is not very helpful, and we will try to avoid it below.

## Geometrical interpretation of scalar triple product 

- The scalar triple product gives the volume of the parallelopiped whose sides are represented by the vectors **a**, **b**, and **c**.




- Vector product $(\mathbf{a} \times \mathbf{b})$ has magnitude equal to the area of the base direction perpendicular to the base.

- The *component* of **c** in this direction is equal to the height of the parallelopiped

  Hence

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = \text{volume of parallelopied}$$

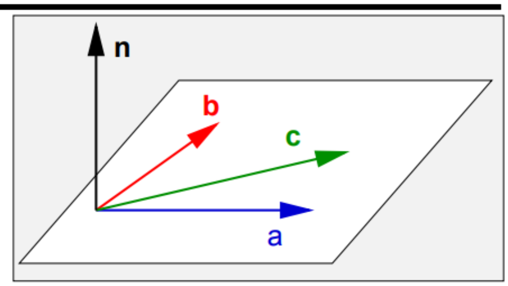
## Linearly dependent vectors 

- If the scalar triple product of three vectors

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

  then the vectors are **linearly dependent**.

$$\mathbf{a} = \lambda \mathbf{b} + \mu \mathbf{c}$$



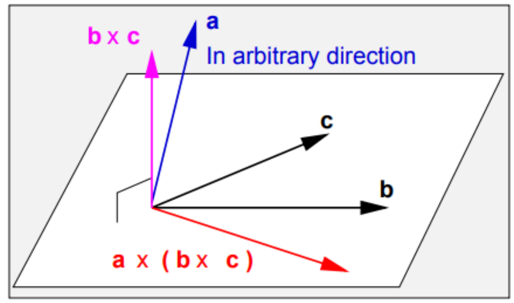

- You can see this immediately either using the determinant

  — The determinant would have one row that was a linear combination of the others

- or geometrically for a 3-dimensional vector.

  — the parallelopiped would have zero volume if squashed flat.

## Vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$




$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to $(\mathbf{b} \times \mathbf{c})$
but $(\mathbf{b} \times \mathbf{c})$ is perpendicular to $\mathbf{b}$ *and* $\mathbf{c}$.
So $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ must be *coplanar* with $\mathbf{b}$ and $\mathbf{c}$.

$$\Rightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \lambda \mathbf{b} + \mu \mathbf{c}$$

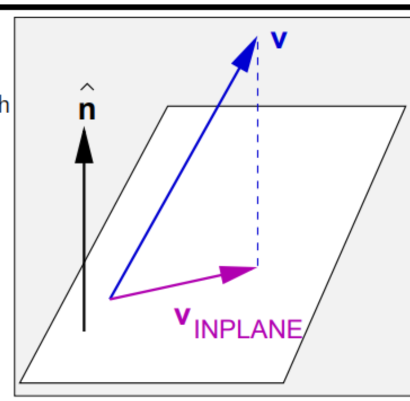
$$\begin{aligned}
(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_1 &= a_2(\mathbf{b} \times \mathbf{c})_3 - a_3(\mathbf{b} \times \mathbf{c})_2 \\
&= a_2(b_1c_2 - b_2c_1) + a_3(b_1c_3 - b_3c_1) \\
&= (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1 \\
&= (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1 \\
&= (\mathbf{a} \cdot \mathbf{c})b_1 - (\mathbf{a} \cdot \mathbf{b})c_1
\end{aligned}$$

Similarly for components 2 and 3: so

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})] = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

## Projection using vector triple product

2.7




- Books say that the vector projection of any old vector $\mathbf{v}$ into a plane with normal $\hat{\mathbf{n}}$ is

$$\mathbf{v}_{\text{INPLANE}} = \hat{\mathbf{n}} \times (\mathbf{v} \times \hat{\mathbf{n}}).$$

- The component of $\mathbf{v}$ in the $\hat{\mathbf{n}}$ direction is $\mathbf{v} \cdot \hat{\mathbf{n}}$
so I would write the vector projection as

$$\mathbf{v}_{\text{INPLANE}} = \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$$

- Can we reconcile the two expressions? Subst. $\hat{\mathbf{n}} \leftarrow \mathbf{a}$, $\mathbf{v} \leftarrow \mathbf{b}$, $\hat{\mathbf{n}} \leftarrow \mathbf{c}$, into our earlier formula

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\
\hat{\mathbf{n}} \times (\mathbf{v} \times \hat{\mathbf{n}}) &= (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\mathbf{v} - (\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}} \\
&= \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}
\end{aligned}$$

- Fantastico! But $\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$ is much easier to understand, cheaper to compute!

## Vector Quadruple Product $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$

- We have just learned that

$$
\begin{aligned}
[\mathbf{p} \times (\mathbf{q} \times \mathbf{r})] &= (\mathbf{p} \cdot \mathbf{r})\mathbf{q} - (\mathbf{p} \cdot \mathbf{q})\mathbf{r} \\
\Rightarrow \quad (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= ??
\end{aligned}
$$

- Regarding $\mathbf{a} \times \mathbf{b}$ as a single vector
  $\Rightarrow$ vqp must be a linear combination of $\mathbf{c}$ and $\mathbf{d}$
- Regarding $\mathbf{c} \times \mathbf{d}$ as a single vector
  $\Rightarrow$ vqp must be a linear combination of $\mathbf{a}$ and $\mathbf{b}$.
- Substituting in carefully (you check ...)

$$
\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\mathbf{d} \\
&= [(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a}]\mathbf{b} - [(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b}]\mathbf{a}
\end{aligned}
$$

## Vector Quadruple Product /ctd

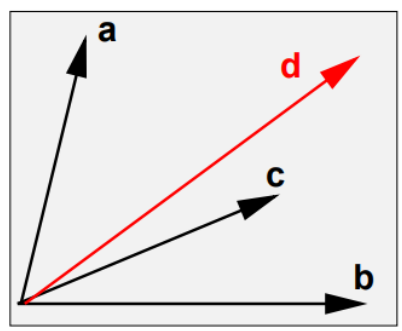
- Using just the R-H sides of what we just wrote ...

$$
[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\,\mathbf{d} = [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\,\mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\,\mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\,\mathbf{c}
$$

- So

$$
\begin{aligned}
\mathbf{d} &= \frac{[(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\,\mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\,\mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\,\mathbf{c}}{[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]} \\
&= \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \quad .
\end{aligned}
$$

- Don't remember by ♡
- **Key point is that the projection of a 3D vector d onto a basis set of 3 non-coplanar vectors is UNIQUE.**



**Question**
Use the quadruple vector product to express the vector $\mathbf{d} = [3, 2, 1]$ in terms of the vectors $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [2, 3, 1]$ and $\mathbf{c} = [3, 1, 2]$.
**Answer**

$$\mathbf{d} = \frac{[(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\, \mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\, \mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\, \mathbf{c}}{[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]}$$

So, grinding away at the determinants, we find

- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -18$ and $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d} = 6$
- $(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d} = -12$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} = -12$.

So

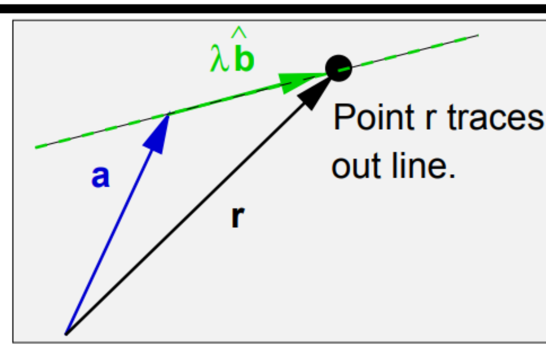
$$\begin{aligned} \mathbf{d} &= \frac{1}{-18}(6\mathbf{a} - 12\mathbf{b} - 12\mathbf{c}) \\ &= \frac{1}{3}(-\mathbf{a} + 2\mathbf{b} + 2\mathbf{c}) \end{aligned}$$

## Geometry using vectors: Lines



- Equation of line passing through point $\mathbf{a}_1$ and lying in the direction of vector $\mathbf{b}$ is

$$\mathbf{r} = \mathbf{a} + \beta \mathbf{b}$$




- **NB! Only when you** make a unit vector in the dirn of $\mathbf{b}$ does the parameter take on the length units defined by $\mathbf{a}$:

$$\mathbf{r} = \mathbf{a} + \lambda \hat{\mathbf{b}}$$

- For a line defined by two points $\mathbf{a}_1$ and $\mathbf{a}_2$

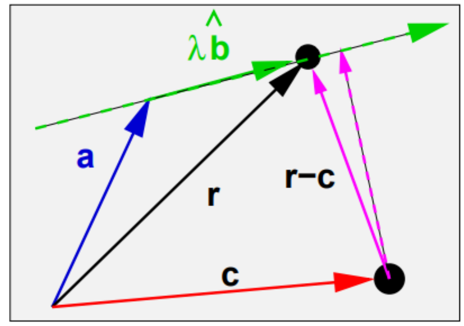
$$\mathbf{r} = \mathbf{a}_1 + \beta(\mathbf{a}_2 - \mathbf{a}_1)$$

- or the unit version ...

$$\mathbf{r} = \mathbf{a}_1 + \lambda(\mathbf{a}_2 - \mathbf{a}_1)/|\mathbf{a}_2 - \mathbf{a}_1|$$

## The shortest distance from a point to a line

2.12




- Vector $\mathbf{p}$ from $\mathbf{c}$ to ANY line point $\mathbf{r}$ is

$$\mathbf{p} = \mathbf{r} - \mathbf{c} = \mathbf{a} + \lambda\hat{\mathbf{b}} - \mathbf{c} = (\mathbf{a} - \mathbf{c}) + \lambda\hat{\mathbf{b}}$$

which has length squared

$$p^2 = (\mathbf{a} - \mathbf{c})^2 + \lambda^2 + 2\lambda(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}} .$$

- Easier to minimize $p^2$ rather than $p$ itself.

$$\frac{d}{d\lambda}p^2 = 0 \quad \text{when} \quad \lambda = -(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}} .$$

- So the minimum length vector is $\mathbf{p} = (\mathbf{a} - \mathbf{c}) - ((\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}$.
  No surprise! It's the component of $(\mathbf{a} - \mathbf{c})$ **perpendicular** to $\hat{\mathbf{b}}$.

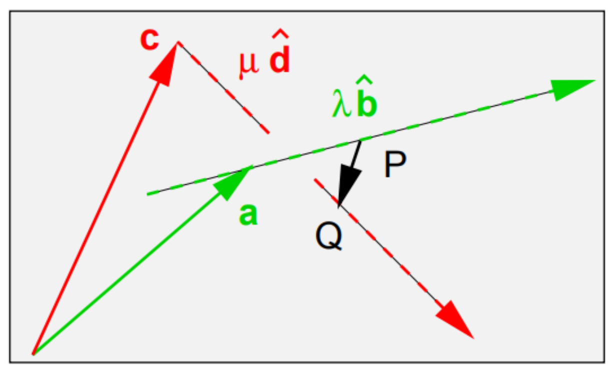
- We could therefore write using the "book" formula ...

$$\begin{aligned} \mathbf{p} &= \hat{\mathbf{b}} \times [(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}] \\ \Rightarrow p_{\min} &= |\hat{\mathbf{b}} \times [(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}]| = |(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}| \quad . \end{aligned}$$

## Shortest distance between two straight lines

2.13

- Shortest distance from point to line is along the perp line
- $\Rightarrow$ shortest distance between two straight lines is along mutual perpendicular.




- The lines are:
  $\mathbf{r} = \mathbf{a} + \lambda\hat{\mathbf{b}}$ $\quad$ $\mathbf{r} = \mathbf{c} + \mu\hat{\mathbf{d}}$
- The unit vector along the mutual perp is

$$\hat{\mathbf{p}} = \frac{\hat{\mathbf{b}} \times \hat{\mathbf{d}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|} .$$

(Yes! Don't forget that $\hat{\mathbf{b}} \times \hat{\mathbf{d}}$ is NOT a unit vector.)

- The minimum length is therefore the component of $(\mathbf{a} - \mathbf{c})$ in this direction

$$p_{\min} = \left| (\mathbf{a} - \mathbf{c}) \cdot \left( \frac{\hat{\mathbf{b}} \times \hat{\mathbf{d}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|} \right) \right| .$$

**Example : the functions** $xk$

Suppose that $f(x)=xk$, where $x>0$ and $k$ is an integer. Then

$$(If)(x)=xk+1k+1,(I2f)(x)=xk+2(k+1)(k+2),$$

and, more generally,

$$(Inf)(x)=k!(n+k)!xn+k=\Gamma(k+1)\Gamma(n+k+1)xn+k.(2.2)$$

Suppose now that $k$ is not a positive integer. Then we still have

$$(Inf)(x)=1(k+1)(k+2)\cdots(n+k)xn+k=\Gamma(k+1)\Gamma(n+k+1)xn+k.$$

We have now shown that (2.2) holds whenever $n$ is a positive integer. "
**Repeated integrals**

Given a function $f(x)$ defined when $x>0$, we can form the indefinite integral of $f$ from $0$ to $x$, and we call this $(If)(x)$; thus

$$(If)(x)=\int x0f(t)dt.$$

If we repeat this process we get the 'second integral'

$$(I2f)(x)=\int x0(If)(t)dt=\int x0(\int t0f(s)ds)dt,$$

and another integration gives the 'third integral'

$$(I3f)(x)=\int x0[\int t0(\int s0f(u)du)ds]dt.(2.1)$$

This looks very complicated (and the formula for the $n$-th integral looks even more complicated), so it is a good idea to look at some simple cases. " **Example : the functions** $xk$

Suppose that $f(x)=x^k$, where $x>0$ and $k$ is an integer. Then

$$(If)(x)=\frac{x^{k+1}}{k+1},\quad (I^2f)(x)=\frac{x^{k+2}}{(k+1)(k+2)},$$

and, more generally,

$$(I^nf)(x)=\frac{k!}{(n+k)!}x^{n+k}=\frac{\Gamma(k+1)}{\Gamma(n+k+1)}x^{n+k}. \quad (2.2)$$

Suppose now that $k$ is not a positive integer. Then we still have

$$(I^nf)(x)=\frac{1}{(k+1)(k+2)\cdots(n+k)}x^{n+k}=\frac{\Gamma(k+1)}{\Gamma(n+k+1)}x^{n+k}.$$

We have now shown that (2.2) holds whenever $n$ is a positive integer. "

**Cauchy's result**

It was Cauchy who showed us how we can look at integrals such as (2.1) in a simpler way, and he showed how we can reduce the $n$ repeated integrals in (2.1) to just one integral. To be precise, he showed that

$$(I^nf)(x)=\frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}f(t)\,dt. \quad (2.3)$$

There is nothing to prove here when $n=1$ because with $n=1$, (2.3) becomes

$$(If)(x)=\frac{1}{0!}\int_0^x (x-t)^0 f(t)\,dt$$

which is just the definition of $(If)(x)$. We shall now prove (2.3) when $n=2$. Let

$$g(x)=\int_0^x (x-t)f(t)\,dt\ ; \quad (2.4)$$

this is the right handside of (2.3) when $n=2$ so we want to show that $g(x)=(I^2f)(x)$. Observe that

$$g(x)=x\int_0^x f(t)\,dt-\int_0^x tf(t)\,dt, \quad (2.5)$$

and if we differentiate both sides of this equation with respect to $x$ (and use the product formula for the first term) we get

$$g'(x)=\left[\int_0^x f(t)\,dt+xf(x)\right]-xf(x)=\int_0^x f(t)\,dt=(If)(x).$$

Now (2.4) implies that $g(0)=0$, so we now have

$$g(x)=g(x)-g(0)=\int_0^x g'(t)\,dt=\int_0^x (If)(t)\,dt=(I^2f)(x)$$

as required. The proof for a general $n$ is similar. We expand the term $(x-t)^{n-1}$ by the Binomial Theorem, and then write $g(x)$ in the manner of (2.5) with all the terms $x^j$ outside the integral sign. The argument then goes as before, and we shall now assume that (2.3) is true for every positive integer $n$. "

## Fractional integrals

The question now is what is $(I^\alpha f)(x)$ when $\alpha$ is any positive number? Following exactly the same idea that we used for the factorial function, we now use Cauchy's formula (2.3) as the basis for our definition of $(I^\alpha f)(x)$. In fact, for every positive $\alpha$ we DEFINE

$$(I^\alpha f)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt.$$

We recall from the previous article that if $\alpha$ is a positive integer, then $\Gamma(\alpha)=(\alpha-1)!$ so this definition of $(I^\alpha f)(x)$ agrees with (2.3) when $\alpha$ is a positive integer. "

## Example : the functions $x^k$ again

Let us now see what $(I^a f)(x)$ is when $f(x)=x^k$ and $a$ is any positive number. Our definition implies that

$$(I^a f)(x)=\frac{1}{\Gamma(a)}\int_0^x (x-t)^{a-1}t^k\,dt,$$

and if we now make the substitution $u=t/x$, we obtain

$$(I^a f)(x)=\frac{x^{a+k}}{\Gamma(a)}\int_0^1 u^k(1-u)^{a-1}\,du.$$

We now have another problem, for there is no simple way to evaluate this definite integral. In fact, many people have studied this integral at great length and, rather remarkably, it turns out to be very closely related to the Gamma function. In fact, if we write

$$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

(this is called the Beta function), where $x$ and $y$ are positive, then we get

$$\mathrm{B}(x,y)=\Gamma(x)\Gamma(y)\Gamma(x+y).$$

Using this, we now see that

$$(I_a f)(x)=x_{a+k}\Gamma(a)B(k+1,a)=x_{a+k}\Gamma(a)(\Gamma(k+1)\Gamma(a)\Gamma(a+k+1))=\Gamma(k+1)\Gamma(a+k+1)x_{a+k},$$

which agrees with (2.2) in the case when $a$ is an integer. In conclusion, we have now shown that if $f(x)=x_k$, and if $x>0$and $a>0$, then

$$(I_a f)(x)=\Gamma(k+1)\Gamma(a+k+1)x_{a+k}.$$

**Example 1** Let us evaluate $(I_{1/2}f)(x)$ when $f(x)=x\sqrt{}=x_{1/2}$. According to the formula, we have

$$(I_{1/2}f)(x)=\Gamma(3/2)\Gamma(2)x=\Gamma(3/2)x=12\Gamma(1/2)x=\pi\sqrt{}2x.$$

**Example 2** Show that with $f(x)=x_2$,

$$(I_{3/2}f)(x)=32105\pi\sqrt{}x_{7/2}.$$

"

### Repeated integration again

Suppose that $f(x)=x_k$, and that $a$ and $b$ are positive. Then

$$(I_b f)(x)=\Gamma(k+1)\Gamma(b+k+1)x_{b+k}=Ag(x),$$

say, where $g(x)=x_{b+k}$. This gives

$$I_a(I_b f)(x)=A\times(I_a g)(x)=\Gamma(k+1)\Gamma(b+k+1)\times\Gamma(b+k+1)\Gamma(a+b+k+1)x_{a+b+k}=(I_{a+b}f)(x).$$

We have now shown that if $f$ is any power of $x$, then

$$(I_a(I_b f))(x)=(I_{a+b}f)(x)=(I_b(I_a f))(x).$$

In fact, this holds for all functions $f$ but this is not easy to prove. Indeed, we shall show in the next article that the corresponding result does NOT hold for fractional derivatives.

# The Unit Tangent and the Unit Normal Vectors

### The Unit Tangent Vector

The derivative of a vector valued function gives a new vector valued function that is tangent to the defined curve. The analogue to the slope of the tangent line is the direction of the tangent line. Since a vector contains a magnitude and a direction, the velocity vector contains more information than we need. We can strip a vector of its magnitude by dividing by its magnitude.

## Definition of the Unit Tangent Vector

Let $\mathbf{r}(t)$ be a differentiable vector valued function and $\mathbf{v}(t) = \mathbf{r}'(t)$ be the velocity vector. Then we define the *unit tangent vector* by as the unit vector in the direction of the velocity vector.

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{||\mathbf{v}(t)||}$$

### Example

Let

$$\mathbf{r}(t) = t\,\mathbf{i} + e^t\,\mathbf{j} - 3t^2\,\mathbf{k}$$

Find the $\mathbf{T}(t)$ and $\mathbf{T}(0)$.

### Solution

We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + e^t\,\mathbf{j} - 6t\,\mathbf{k}$$

and

$$\| v(t) \| = \sqrt{1+e^{2t}+36t^2}$$

To find the unit tangent vector, we just divide

$$T(t) = \frac{v(t)}{\| v(t) \|} = \frac{i+e^t j - 6tk}{\sqrt{1+e^{2t}+36t^2}}$$

To find T(0) plug in 0 to get

$$T(0) = \frac{i+e^0 j - 6(0)k}{\sqrt{1+e^{2(0)}+36(0)^2}} = \frac{i+j}{\sqrt{2}} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$$

---

### The Principal Unit Normal Vector

A normal vector is a perpendicular vector. Given a vector **v** in the space, there are infinitely many perpendicular vectors. Our goal is to select a special vector that is normal to the unit tangent vector. Geometrically, for a non straight curve, this vector is the unique vector that point into the curve. Algebraically we can compute the vector using the following definition.

## Definition of the Principal Unit Normal Vector

Let **r**(t) be a differentiable vector valued function and let **T**(t) be the unit tangent vector. Then the *principal unit normal vector* **N**(t) is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\quad}$$

$$||\mathbf{T}'(t)||$$

Comparing this with the formula for the unit tangent vector, if we think of the unit tangent vector as a vector valued function, then the principal unit normal vector is the unit tangent vector of the unit tangent vector function. You will find that finding the principal unit normal vector is almost always cumbersome. The quotient rule usually rears its ugly head.

**Example**

Find the unit normal vector for the vector valued function

$$\mathbf{r}(t) = t\mathbf{i} + t^2\,\mathbf{j}$$

and sketch the curve, the unit tangent and unit normal vectors when $t = 1$.

**Solution**

First we find the unit tangent vector

$$T(t) = \frac{i + 2tj}{\sqrt{1+4t^2}}$$

Now use the quotient rule to find **T**'(t)

$$T'(t) = \frac{(1+4t^2)^{1/2}(2j) - (i+2tj)4t(1+4t^2)^{-1/2}}{1+4t^2}$$

Since the unit vector in the direction of a given vector will be the same after multiplying the vector by a positive scalar, we can simplify by multiplying by the factor

$$(1+4t^2)(1+4t^2)^{1/2}$$

The first factor gets rid of the denominator and the second factor gets rid of the fractional power. We have

$$T'(t)(1+4t^2)(1+4t^2)^{1/2} = (1+4t^2)(2j) - (i+2tj)4t = -4ti + 2j$$

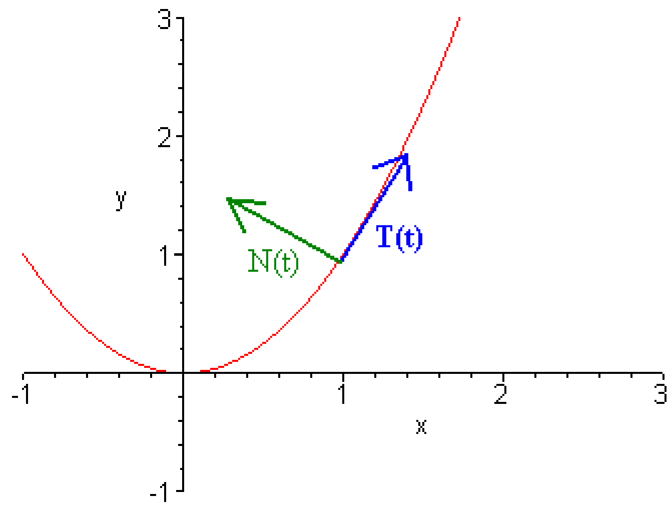
Now we divide by the magnitude (after first dividing by 2) to get

$$N(t) = \frac{-2ti + j}{\sqrt{1+4t^2}}$$

Now plug in 1 for both the unit tangent vector to get

$$T(1) = \frac{1}{\sqrt{5}}i + \frac{2}{\sqrt{5}}j \qquad N(1) = -\frac{2}{\sqrt{5}}i + \frac{1}{\sqrt{5}}j$$

The picture below shows the graph and the two vectors.




## Tangential and Normal Components of Acceleration

Imagine yourself driving down from Echo Summit towards Myers and having your brakes fail. As you are riding you will experience two forces (other than the force of terror) that will change the velocity. The force of gravity will cause the car to increase in speed. A second change in velocity will be caused by the car going around the curve. The first component of acceleration is called the *tangential component* of acceleration and the second is called the *normal component of acceleration*. As you may guess the tangential component of acceleration is in the direction of the unit tangent vector and the normal component of acceleration is in the direction of the principal unit normal vector. Once we have **T** and **N**, it is straightforward to find the two components. We have

## Tangential and Normal Components of Acceleration

The tangential component of acceleration is

$$a_T = a \bullet T = \frac{v \bullet a}{||v||}$$

and the normal component of acceleration is

$$a_N = a \bullet N = \frac{||v \times a||}{||v||}$$

and

$$\mathbf{a} = \mathbf{a_N N} + \mathbf{a_T T}$$

**Proof**

First notice that

$$\mathbf{v} = ||\mathbf{v}||\,\mathbf{T} \qquad \text{and} \qquad \mathbf{T'} = ||\mathbf{T'}||\,\mathbf{N}$$

Taking the derivative of both sides gives

$$\mathbf{a} = \mathbf{v'} = ||\mathbf{v}||'\,\mathbf{T} + ||\mathbf{v}||\,\mathbf{T'} = ||\mathbf{v}||'\,\mathbf{T} + ||\mathbf{v}||\,||\mathbf{T'}||\,\mathbf{N}$$

This tells us that the acceleration vector is in the plane that contains the unit tangent vector and the unit normal vector. The first equality follows immediately from the definition of the component of a vector in the direction of another vector. The second equalities will be left as exercises.

**Example**

Find the tangential and normal components of acceleration for the prior example

$$\mathbf{r}(t) = t\mathbf{i} + t^2\,\mathbf{j}$$

**Solution**

Taking two derivatives, we have

$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{j}$$

We dot the acceleration vector with the unit tangent and normal vectors to get

$$a_T(t) = a \bullet T = \frac{4t}{\sqrt{1+4t^2}}$$

$$a_N(t) = a \bullet N = \frac{2}{\sqrt{1+4t^2}}$$

# Vector Operators: Grad, Div and Curl

In the first lecture of the second part of this course we move more to consider properties of fields. We introduce three field operators which reveal interesting collective field properties, viz.

- the **gradient** of a scalar field,
- the **divergence** of a vector field, and
- the **curl** of a vector field.

There are two points to get over about each:

- The mechanics of taking the grad, div or curl, for which you will need to brush up your multivariate calculus.
- The underlying physical meaning — that is, why they are worth bothering about.

In Lecture 6 we will look at combining these vector operators.

## 5.1 The gradient of a scalar field

Recall the discussion of temperature distribution throughout a room in the overview, where we wondered how a scalar would vary as we moved off in an arbitrary direction. Here we find out how.

If $U(\mathbf{r}) = U(x, y, z)$ is a scalar field, ie a scalar function of position $\mathbf{r} = [x, y, z]$ in 3 dimensions, then its **gradient** at any point is defined in Cartesian co-ordinates by

$$\text{grad}U = \frac{\partial U}{\partial x}\hat{\mathbf{\imath}} + \frac{\partial U}{\partial y}\hat{\mathbf{\jmath}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \, . \tag{5.1}$$

It is usual to define the **vector operator** which is called "del" or "nabla"

$$\nabla = \hat{\mathbf{i}}\,\frac{\partial}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\,. \tag{5.2}$$

Then

$$\mathrm{grad}U \equiv \nabla U\,. \tag{5.3}$$

**Note immediately that $\nabla U$ is a vector field!**

Without thinking too carefully about it, we can see that the gradient of a scalar field tends to point in the direction of greatest change of the field. Later we will be more precise.

### ♣ Worked examples of gradient evaluation

1. $U = x^2$

$$\Rightarrow \nabla U = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) x^2 = 2x\hat{\mathbf{i}}\,. \tag{5.4}$$

2. $U = r^2$

$$r^2 = x^2 + y^2 + z^2 \tag{5.5}$$

$$\Rightarrow \nabla U = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)(x^2 + y^2 + z^2) \tag{5.6}$$

$$= 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}} = 2\,\mathbf{r}\,. \tag{5.7}$$

3. $U = \mathbf{c}\cdot\mathbf{r}$, where $\mathbf{c}$ is constant.

$$\Rightarrow \nabla U = \left(\hat{\mathbf{i}}\,\frac{\partial}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right)(c_1 x + c_2 y + c_3 z) = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}} = \mathbf{c}\,. \tag{5.8}$$

4. $U = U(r)$, where $r = \sqrt{(x^2 + y^2 + z^2)}$. **NB NOT** $U(\mathbf{r})$.

   $U$ is a function of $r$ alone so $dU/dr$ exists. As $U = U(x, y, z)$ also,

$$\frac{\partial U}{\partial x} = \frac{dU}{dr}\frac{\partial r}{\partial x} \qquad \frac{\partial U}{\partial y} = \frac{dU}{dr}\frac{\partial r}{\partial y} \qquad \frac{\partial U}{\partial z} = \frac{dU}{dr}\frac{\partial r}{\partial z}\,. \tag{5.9}$$

$$\Rightarrow \nabla U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} = \frac{dU}{dr}\left(\frac{\partial r}{\partial x}\hat{\mathbf{i}} + \frac{\partial r}{\partial y}\hat{\mathbf{j}} + \frac{\partial r}{\partial z}\hat{\mathbf{k}}\right) \tag{5.10}$$

But $r = \sqrt{x^2 + y^2 + z^2}$, so $\partial r/\partial x = x/r$ and similarly for $y, z$.

$$\Rightarrow \nabla U = \frac{dU}{dr}\left(\frac{x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}}{r}\right) = \frac{dU}{dr}\left(\frac{\mathbf{r}}{r}\right) \quad . \tag{5.11}$$

## 5.2 The significance of grad

If our current position is $\mathbf{r}$ in some scalar field $U$ (Fig. 5.1(a)), and we move an infinitesimal distance $d\mathbf{r}$, we know that the change in $U$ is

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \ . \tag{5.12}$$

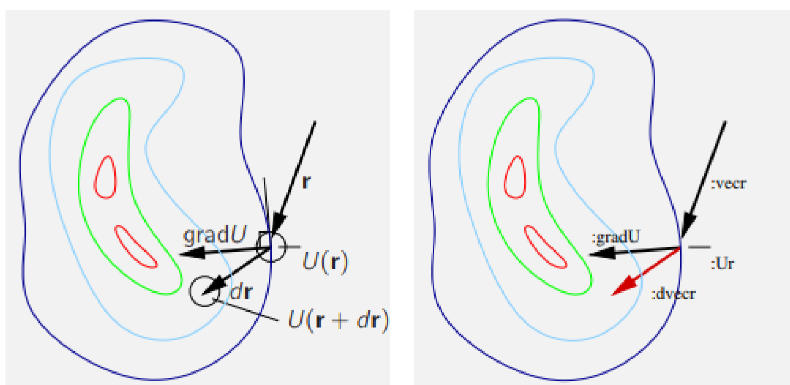
But we know that $d\mathbf{r} = (\hat{\mathbf{\imath}}dx + \hat{\mathbf{\jmath}}dy + \hat{\mathbf{k}}dz)$ and $\nabla U = (\hat{\mathbf{\imath}}\partial U/\partial x + \hat{\mathbf{\jmath}}\partial U/\partial y + \hat{\mathbf{k}}\partial U/\partial z)$, so that the change in $U$ is also given by the scalar product

$$dU = \nabla U \cdot d\mathbf{r} \ . \tag{5.13}$$

Now divide both sides by $ds$

$$\frac{dU}{ds} = \nabla U \cdot \frac{d\mathbf{r}}{ds} \ . \tag{5.14}$$

But remember that $|d\mathbf{r}| = ds$, so $d\mathbf{r}/ds$ is a unit vector in the direction of $d\mathbf{r}$.




(a) (b)

Figure 5.1: The directional derivative: The rate of change of $U$ wrt distance in direction $\hat{\mathbf{d}}$ is $\nabla U \cdot \hat{\mathbf{d}}$.

This result can be paraphrased (Fig. 5.1(b)) as:

- grad$U$ has the property that the rate of change of $U$ wrt distance in a particular direction ($\hat{\mathbf{d}}$) is the projection of grad$U$ onto that direction (or the component of grad$U$ in that direction).

The quantity $dU/ds$ is called a **directional derivative**, but note that in general it has a different value for each direction, and so has no meaning until you specify the direction.

We could also say that

- At any point P, grad$U$ points in the direction of greatest change of $U$ at P, and has magnitude equal to the rate of change of $U$ wrt distance in that direction.

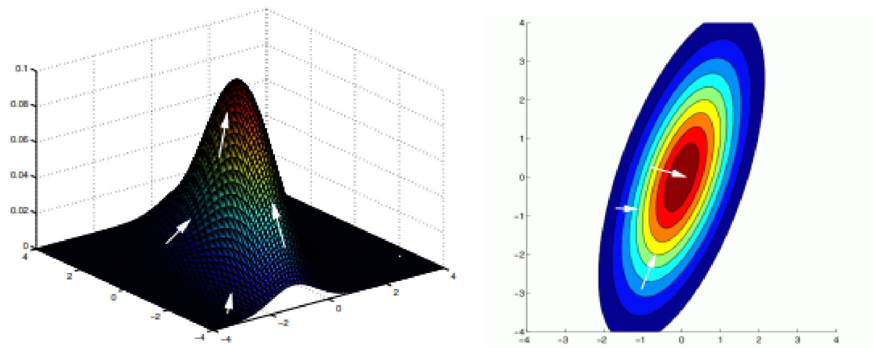


Figure 5.2: $\nabla U$ is in the direction of greatest (positive!) change of $U$ wrt distance. (Positive $\Rightarrow$"uphill".)

Another nice property emerges if we think of a surface of constant $U$ – that is the locus $(x, y, z)$ for $U(x, y, z) =$ constant. If we move a tiny amount within that iso-$U$ surface, there is no change in $U$, so $dU/ds = 0$. So for any $d\mathbf{r}/ds$ in the surface

$$\nabla U \cdot \frac{d\mathbf{r}}{ds} = 0 . \tag{5.15}$$

But $d\mathbf{r}/ds$ is a tangent to the surface, so this result shows that

- grad$U$ is everywhere NORMAL to a surface of constant $U$.

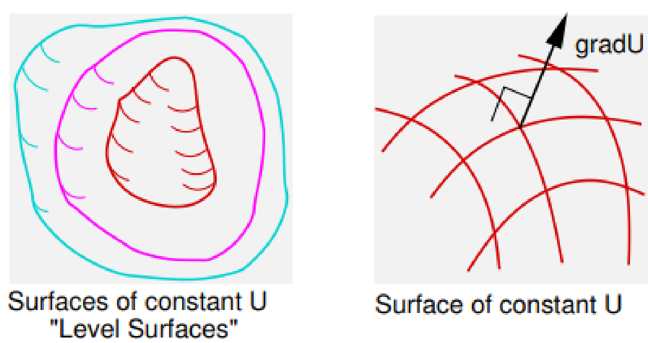



Figure 5.3: grad$U$ is everywhere NORMAL to a surface of constant $U$.

The divergence computes a scalar quantity from a vector field by differentiation.

If $\mathbf{a}(x, y, z)$ is a vector function of position in 3 dimensions, that is $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, then its divergence at any point is defined in Cartesian co-ordinates by

$$\operatorname{div}\mathbf{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \tag{5.16}$$

We can write this in a simplified notation using a scalar product with the $\nabla$ vector differential operator:

$$\operatorname{div}\mathbf{a} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot\mathbf{a} = \nabla\cdot\mathbf{a} \tag{5.17}$$

Notice that the divergence of a vector field is a scalar field.

### ♣ Examples of divergence evaluation

| | $\mathbf{a}$ | $\operatorname{div}\mathbf{a}$ |
|---|---|---|
| 1) | $x\hat{\mathbf{i}}$ | 1 |
| 2) | $\mathbf{r}(= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$ | 3 |
| 3) | $\mathbf{r}/r^3$ | 0 |
| 4) | $r\mathbf{c}$, for $\mathbf{c}$ constant | $(\mathbf{r}\cdot\mathbf{c})/r$ |

We work through example 3).

The $x$ component of $\mathbf{r}/r^3$ is $x.(x^2+y^2+z^2)^{-3/2}$, and we need to find $\partial/\partial x$ of it.

$$\begin{aligned}\frac{\partial}{\partial x}x.(x^2+y^2+z^2)^{-3/2} &= 1.(x^2+y^2+z^2)^{-3/2} + x\frac{-3}{2}(x^2+y^2+z^2)^{-5/2}.2x \\ &= r^{-3}\left(1-3x^2r^{-2}\right)\ . \end{aligned} \tag{5.18}$$

The terms in $y$ and $z$ are similar, so that

$$\begin{aligned}\operatorname{div}(\mathbf{r}/r^3) &= r^{-3}\left(3-3(x^2+y^2+z^2)r^{-2}\right) = r^{-3}(3-3) \\ &= 0\end{aligned} \tag{5.19}$$

## 5.4 The significance of div

Consider a typical vector field, water flow, and denote it by $\mathbf{a}(\mathbf{r})$. This vector has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of $\mathbf{a}$ per unit time.

Now take an infinitesimal volume element $dV$ and figure out the balance of the flow of $\mathbf{a}$ in and out of $dV$.

To be specific, consider the volume element $dV = dxdydz$ in Cartesian co-ordinates, and think first about the face of area $dxdz$ perpendicular to the $y$ axis and facing outwards in the negative $y$ direction. (That is, the one with surface area $d\mathbf{S} = -dxdz\hat{\mathbf{j}}$.)

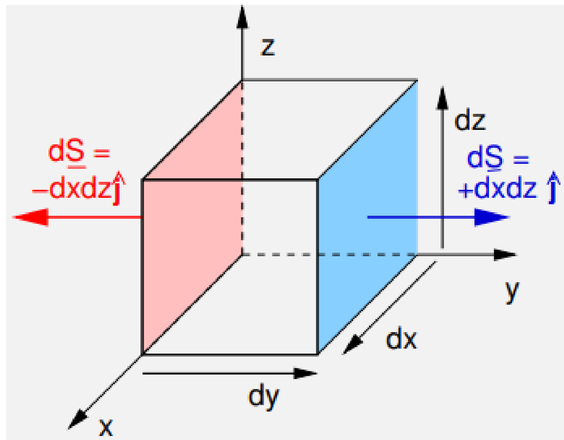



Figure 5.4: Elemental volume for calculating divergence.

The component of the vector $\mathbf{a}$ normal to this face is $\mathbf{a} \cdot \hat{\mathbf{j}} = a_y$, and is pointing inwards, and so the its contribution to the OUTWARD flux from this surface is

$$\mathbf{a} \cdot d\mathbf{S} \quad = \quad -a_y(x, y, z)dzdx \; , \tag{5.20}$$

(By the way, flux here denotes mass per unit time.)

A similar contribution, but of opposite sign, will arise from the opposite face, but we must remember that we have moved along $y$ by an amount $dy$, so that this OUTWARD amount is

$$a_y(x, y + dy, z)dzdx = \left( a_y + \frac{\partial a_y}{\partial y} dy \right) dxdz \tag{5.21}$$

The total outward amount from these two faces is

$$\frac{\partial a_y}{\partial y} dydxdz = \frac{\partial a_y}{\partial y} dV \tag{5.22}$$

Summing the other faces gives a total outward flux of

$$\left( \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) dV = \nabla \cdot \mathbf{a} \;\; dV \tag{5.23}$$

So we see that

$$\boxed{\text{The divergence of a vector field represents the flux generation per unit volume at each point of the field. (Divergence because it is an efflux not an influx.)}}$$

Interestingly we also saw that the total efflux from the infinitesimal volume was equal to the flux integrated over the surface of the volume.

(NB: The above does not constitute a rigorous proof of the assertion because we have not proved that the quantity calculated is independent of the co-ordinate system used, but it will suffice for our purposes.)

## 5.5 The Laplacian: $\mathrm{div}\,(\mathrm{grad}U)$ of a scalar field

Recall that $\mathrm{grad}U$ of *any* scalar field $U$ is a vector field. Recall also that we can compute the divergence of any vector field. So we can certainly compute $\mathrm{div}\,(\mathrm{grad}U)$, even if we don't know what it means yet.

Here is where the $\nabla$ operator starts to be really handy.

$$\begin{aligned}
\nabla\cdot(\nabla U) &= \left(\hat{\mathbf{\imath}}\frac{\partial}{\partial x}+\hat{\mathbf{\jmath}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot\left(\left(\hat{\mathbf{\imath}}\frac{\partial}{\partial x}+\hat{\mathbf{\jmath}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)U\right) && (5.24)\\
&= \left(\left(\hat{\mathbf{\imath}}\frac{\partial}{\partial x}+\hat{\mathbf{\jmath}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\mathbf{\imath}}\frac{\partial}{\partial x}+\hat{\mathbf{\jmath}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\right)U && (5.25)\\
&= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)U && (5.26)\\
&= \left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right) && (5.27)\\
& && (5.28)
\end{aligned}$$

This last expression occurs frequently in engineering science (you will meet it next in solving Laplace's Equation in partial differential equations). For this reason, the operator $\nabla^2$ is called the "Laplacian"

$$\nabla^2 U = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)U \tag{5.29}$$

Laplace's equation itself is

$$\nabla^2 U = 0 \tag{5.30}$$

♣ **Examples of $\nabla^2 U$ evaluation**

| | $U$ | $\nabla^2 U$ |
|---|---|---|
| 1) | $r^2 (= x^2 + y^2 + z^2)$ | $6$ |
| 2) | $xy^2z^3$ | $2xz^3 + 6xy^2z$ |
| 3) | $1/r$ | $0$ |

Let's prove example (3) (which is particularly significant – can you guess why?).

$$1/r = (x^2 + y^2 + z^2)^{-1/2} \tag{5.31}$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-1/2} = \frac{\partial}{\partial x} - x.(x^2 + y^2 + z^2)^{-3/2} \tag{5.32}$$

$$= -(x^2 + y^2 + z^2)^{-3/2} + 3x.x.(x^2 + y^2 + z^2)^{-5/2} \tag{5.33}$$

$$= (1/r^3)(-1 + 3x^2/r^2) \tag{5.34}$$

Adding up similar terms for $y$ and $z$

$$\nabla^2 \frac{1}{r} = \frac{1}{r^3}\left(-3 + 3\frac{(x^2 + y^2 + x^2)}{r^2}\right) = 0 \tag{5.35}$$

## 5.6 The curl of a vector field

So far we have seen the operator $\nabla$ applied to a scalar field $\nabla U$; and dotted with a vector field $\nabla \cdot \mathbf{a}$.

We are now overwhelmed by an irrestible temptation to

- cross it with a vector field $\nabla \times \mathbf{a}$

This gives the **curl of a vector field**

$$\nabla \times \mathbf{a} \equiv \text{curl}(\mathbf{a}) \tag{5.36}$$

We can follow the pseudo-determinant recipe for vector products, so that

$$\nabla \times \mathbf{a} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \quad \text{(remember it this way)} \tag{5.37}$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{\imath}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{\jmath}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{k}} \tag{5.38}$$

♣ **Examples of curl evaluation**

| | $\mathbf{a}$ | $\nabla \times \mathbf{a}$ |
|---|---|---|
| 1) | $-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ | $2\hat{\mathbf{k}}$ |
| 2) | $x^2y^2\hat{\mathbf{k}}$ | $2x^2y\hat{\mathbf{i}} - 2xy^2\hat{\mathbf{j}}$ |

## 5.7 The signficance of curl

Perhaps the first example gives a clue. The field $\mathbf{a} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ is sketched in Figure 5.5. (It is the field you would calculate as the velocity field of an object rotating with $\boldsymbol{\omega} = [0, 0, 1]$.) This field has a curl of $2\hat{\mathbf{k}}$, which is in the r-h screw sense out of the page. You can also see that a field like this must give a finite value to the line integral around the complete loop $\oint_C \mathbf{a} \cdot d\mathbf{r}$.

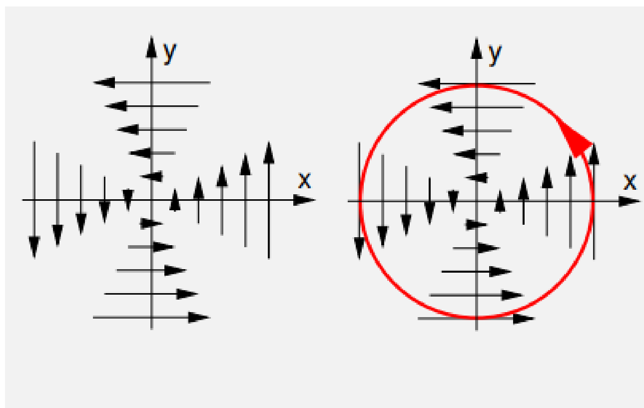



Figure 5.5: A rough sketch of the vector field $-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$.

In fact curl is closely related to the line integral around a loop.

> The **circulation** of a vector $\mathbf{a}$ round any closed curve $C$ is defined to be $\oint_C \mathbf{a} \cdot d\mathbf{r}$ and the **curl** of the vector field $\mathbf{a}$ represents the **vorticity**, or **circulation per unit area**, of the field.

Our proof uses the small rectangular element $dx$ by $dy$ shown in Figure 5.6.

Consider the circulation round the perimeter of a rectangular element.

The fields in the $x$ direction at the bottom and top are

$$a_x(x, y, z) \qquad \text{and} \qquad a_x(x, y + dy, z) = a_x(x, y, z) + \frac{\partial a_x}{\partial y} dy, \tag{5.39}$$

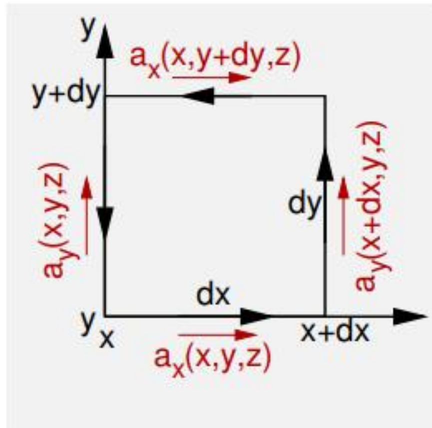



Figure 5.6: A small element used to calculate curl.

and the fields in the $y$ direction at the left and right are

$$a_y(x, y, z) \qquad \text{and} \qquad a_y(x + dx, y, z) = a_y(x, y, z) + \frac{\partial a_y}{\partial x}dx \tag{5.40}$$

Starting at the bottom and working round in the anticlockwise sense, the four contributions to the circulation $dC$ are therefore as follows, where the minus signs take account of the path being opposed to the field:

$$\begin{aligned} dC &= +[a_x\ dx] + [a_y(x + dx, y, z)\ dy] - [a_x(x, y + dy, z)\ dx] - [a_y\ dy] \quad (5.41) \\ &= +[a_x\ dx] + \left[\left(a_y + \frac{\partial a_y}{\partial x}dx\right)\ dy\right] - \left[\left(a_x(y) + \frac{\partial a_x}{\partial y}dy\right)\ dx\right] - [a_y\ dy] \\ &= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\ dx\ dy \\ &= (\nabla \times \mathbf{a}) \cdot d\mathbf{S} \end{aligned}$$

where $d\mathbf{S} = dxdy\hat{\mathbf{k}}$.

**NB:** Again, this is not a completely rigorous proof as we have not shown that the result is independent of the co-ordinate system used.

# Green's, Stokes's, and Gauss's Theorems

Let $D$ be a closed bounded region in $\mathbb{R}^2$ such that its boundary $\partial D$ consists of finitely many simple closed curves that are **oriented** in such a way that $D$ in on the **left** as one traverses $\partial D$. Let $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ be a vector field of class $C^1$.

1. **(Green's Theorem)** $\displaystyle\oint_{\partial D} M\,dx + N\,dy = \iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$

2. **(Vector form of Green's Theorem)** $\displaystyle\oint_{\partial D} \mathbf{F}\cdot d\mathbf{s} = \iint\limits_D (\nabla \times \mathbf{F})\cdot \mathbf{k}\,dA.$

3. **(Divergence Theorem in the plane)** **<u>If</u>** $\mathbf{n}$ is the outward unit normal vector to $D$, **<u>then</u>** $\displaystyle\oint_{\partial D} \mathbf{F}\cdot\mathbf{n}\,ds = \iint\limits_D \nabla\cdot\mathbf{F}\,dA.$

Let $S$ be a bounded, oriented surface in $\mathbb{R}^3$ such that its boundary $\partial S$ consists of finitely many simple closed curves that are oriented **consistently** with $S$. Let $\mathbf{F}$ be a vector field of class $C^1$.

4. **(Stokes's Theorem)** $\displaystyle\oint_{\partial S} \mathbf{F}\cdot d\mathbf{s} = \iint\limits_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}.$

---

Let $W$ be a bounded solid region in $\mathbb{R}^3$ such that its boundary $\partial W$ consists of finitely many closed orientable surfaces that are oriented by unit normals $\mathbf{n}$ pointing **away** from $W$. Let $\mathbf{F}$ be a vector field of class $C^1$.

5. **(Gauss's Theorem)** $\displaystyle\oiint\limits_{\partial W} \mathbf{F}\cdot d\mathbf{S} = \oiint\limits_{\partial W} (\mathbf{F}\cdot\mathbf{n})dS = \iiint\limits_W \nabla\cdot\mathbf{F}\,dV$

# UNIT-2

## Mechanics

### Definition

The branch of physics that deals with the action of forces on bodies and with motion comprised of kinetics, statics and kinematics. It is the study of the action of forces on the body and the corresponding reaction of the body to the environment.

### Types of Mechanics

**Kinematics:** Study of motion of objects without taking into account the factor which causes the motion that is nature of force

**Projectiles:** A particle when thrown into space and moves in two dimensions under the influence of only gravity and constant acceleration is called projectile. The path traversed by the projectile is called trajectory. The trajectory of a projectile which is moving under the influence of a constant acceleration is a parabola

**Circular Motion:** When a particle moves in a plane such that it maintains a constant distance from a fixed or moving point then the motion is said to be a Circular motion with respect to that fixed point.

Uniform and Non-uniform motion:

Relative Velocity:

Newton's Law of Motion:

Law of gravitation

Center of Mass:

Collisions rotational Motion, fluid Mechanics

**Comparison between Uniform and Non-Uniform Circular Motion**

| Uniform Circular Motion | Non-uniform Circular Motion |
|---|---|
| Speed is constant | Speed is variable |
| Angular speed(w) is constant | Angular speed (w) is variable |
| Angular acceleration is zero | Angular acceleration is non-zero |
| Tangential acceleration is zero | Tangential acceleration is non-zero |
| Modulus of acceleration is constant but the vector will be variable | Modulus of acceleration is variable and the vector will also be variable |
| Acceleration is directed always towards the center | Acceleration is directed always away from the center |

## Relative velocity

When you are traveling in a car or bus or train, you see the trees, buildings and many other things outside going backwards. But are they really going backwards? No, you know it pretty well that it's your vehicle that is moving while the trees are stationary on the ground. But then why do the trees appear to be moving backwards? Also the co-passengers with you who are moving appear stationary to you despite moving.

It's because in your frame both you and your co-passengers are moving together. Which means there is no relative velocity between you and the passengers.Whereas the trees are stationary while you are moving. Therefore trees are moving at some relative velocity with respect to you and the other passenger. And that relative velocity is the difference of velocities between you and the tree.

The relative velocity is the velocity of an object or observer B in the rest frame of another object or the observer A. The general formula of velocity is : **Velocity of B relative to A is =** $\vec{v}_b - \vec{v}_a$

This is the only formula that describes the concept of relative velocity. When two objects are moving in the same direction, then

$$\vec{v}_{ab} = \vec{v}_a + \vec{v}_b$$

When two objects are moving in the opposite direction, then

$$\vec{v}_{ab} = \vec{v}_a - \vec{v}_b$$

Lets us understand the concept of relative velocity with this example.

Consider two trains moving with same speed and in the same direction. Even if both the trains are in motion with respect to buildings, trees along the two sides of the track, yet to the observer of the train, the other train does not seem to be moving at all. the velocity of the train appears to be zero.

Suppose you are in a car moving at 50mph.The 50 mph is your relative velocity as compared to the surface of the earth.At the same time if I am sitting next to you your relative velocity compared to me is zero. If we were on a bus and you walked forward at 1 mph, your relative velocity on the earth would be 51 mph and your relative velocity compared to me would be 1 mph. Relative velocity is simply any objects speed compared to any other object regardless of its speed.

**Newton's First Law of Motion:**

> **I. Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.**

**This we recognize as essentially Galileo's concept of inertia, and this is often termed simply the "Law of Inertia".**

**Newton's Second Law of Motion:**

> **II. The relationship between an object's mass *m*, its acceleration a, and the applied force *F* is *F = ma*. Acceleration and force are vectors (as indicated by their symbols being displayed in slant bold font); in this law the direction of the force vector is the same as the direction of the acceleration vector.**

**This is the most powerful of Newton's three Laws, because it allows quantitative calculations of dynamics: how do velocities change when forces are applied. Notice the fundamental difference between Newton's 2nd Law and the dynamics of Aristotle: according to Newton, a force causes only a *change in velocity* (an acceleration); it does not maintain the velocity as Aristotle held.**

**This is sometimes summarized by saying that under Newton, *F = ma*, but under Aristotle *F = mv*, where *v* is the velocity. Thus, according to Aristotle there is**

**only a velocity if there is a force, but according to Newton an object with a certain velocity maintains that velocity *unless* a force acts on it to cause an acceleration (that is, a change in the velocity). As we have noted earlier in conjunction with the discussion of Galileo, Aristotle's view seems to be more in accord with common sense, but that is because of a failure to appreciate the role played by frictional forces. Once account is taken of *all* forces acting in a given situation it is the dynamics of Galileo and Newton, not of Aristotle, that are found to be in accord with the observations.**

**Newton's Third Law of Motion:**

> **III. For every action there is an equal and opposite reaction.**

**This law is exemplified by what happens if we step off a boat onto the bank of a lake: as we move in the direction of the shore, the boat tends to move in the opposite direction (leaving us facedown in the water, if we aren't careful!).**

# The Four Fundamental Forces of Nature

The Four Fundamental Forces of Nature are Gravitational force, Weak Nuclear force, Electromagnetic force and Strong Nuclear force. The weak and strong forces are effective only over a very short range and dominate only at the level of subatomic particles. Gravity and Electromagnetic force have infinite range. Let's see each of them in detail.

**The Four Fundamental Forces and their strengths**

1. Gravitational Force – Weakest force; but infinite range. (Not part of standard model)
2. Weak Nuclear Force – Next weakest; but short range.
3. Electromagnetic Force – Stronger, with infinite range.
4. Strong Nuclear Force – Strongest; but short range.

### Gravitational Force

The gravitational force is weak, but very long ranged. Furthermore, it is always attractive. It acts between any two pieces of matter in the Universe since mass is its source.

### Weak Nuclear Force

The weak force is responsible for radioactive decay and neutrino interactions. It has a very short range and. As its name indicates, it is very weak. The weak force causes Beta decay ie. the conversion of a neutron into a proton, an electron and an antineutrino.

### Electromagnetic Force

The electromagnetic force causes electric and magnetic effects such as the repulsion between like electrical charges or the interaction of bar magnets. It is long-ranged, but much weaker than the strong force. It can be attractive or repulsive, and acts only between pieces of matter carrying electrical charge. Electricity, magnetism, and light are all produced by this force.

### Strong Nuclear Force

The strong interaction is very strong, but very short-ranged. It is responsible for holding the nuclei of atoms together. It is basically attractive, but can be effectively repulsive in some circumstances. The strong force is ‘carried’ by particles called gluons; that is, when two particles interact through the strong force, they do so by exchanging gluons. Thus, the quarks inside of the protons and neutrons are bound together by the exchange of the strong nuclear force.

**Fundamental Force Particles**

| Force | Particles Experiencing | Force Carrier Particle | Range | Relative Strength* |
|---|---|---|---|---|
| **Gravity** acts between objects with mass | all particles with mass | graviton (not yet observed) | infinity | much weaker |
| **Weak Force** governs particle decay | quarks and leptons | $W^+$, $W^-$, $Z^0$ (W and Z) | short range | |
| **Electromagnetism** acts between electrically charged particles | electrically charged | $\gamma$ (photon) | infinity | |
| **Strong Force**** binds quarks together | quarks and gluons | $g$ (gluon) | short range | much stronger |

## What is Pseudo Force?

A Pseudo force (also called as fictitious force, inertial force or dâ€™Alembert force) is an apparent force that acts on all masses whose motion is described using a non-inertial frame of reference frame, such as rotating reference frame.
Pseudo force comes in effect when the frame of reference has started acceleration compared to a non-accelerating frame.
The force F does not arise from any physical interaction between two objects, but rather from the acceleration â€˜aâ€™ of the non-inertial reference frame itself. As a frame can accelerate in any arbitrary way, so can pseudo forces be as arbitrary (but only in direct response to the acceleration of the frame). However, four pseudo forces are defined for frames accelerated in commonly occurring ways: one by relative acceleration of the origin in a straight line (rectilinear acceleration); two involving rotation: Coriolis force and Centrifugal force and fourth called Euler force, caused by a variable rate of rotation.

## *Examples of Pseudo Force:*

For example if you consider a person standing at a bus stop watching an accelerating car, he infers that a force is exerted on the car and it is accelerating. Here there is no problem and the pseudo force concept is not required
But, if the person inside the accelerating car is looking at the person standing at the bus stop, he finds that the person is accelerating with respect to the car, though no force is acting on it. Here, the concept of pseudo force is required to convert the non-inertial frame of reference to an equivalent inertial frame of reference.

Another example Consider a ball hung from the roof of a train by means of an inextensible string. If the train is at rest or is moving with a uniform speed in a straight line the string will be vertical. A passenger will infer that the net force acting on the ball is zero.
If the train begins to accelerate, then the string will be making an angle with respect to the vertical. For the passenger, there are only two forces and they are not collinear. But, the ball remains apparently in a state of equilibrium (as long as the acceleration of the train is constant). Here, the concept of pseudo force is required.

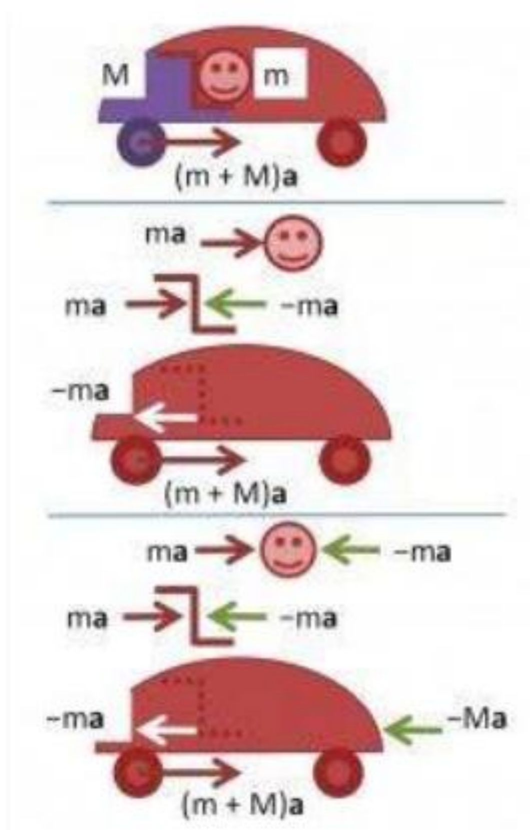



**Figure 1:**

**Top panel:** accelerating car of mass M with passenger of mass m. The force from the axle is (m + M)a. In the inertial frame, this is the only force on the car and passenger.

**Center panel:** an exploded view in the inertial frame. The passenger is subject to the accelerating force ma. The seat (assumed of negligible mass) is compressed between the reaction force â€“ma and the applied force from the car ma. The car is subject to the net acceleration force Ma that is the difference between the applied force (m + M)a from the axle and the reaction from the seat âˆ’ma.

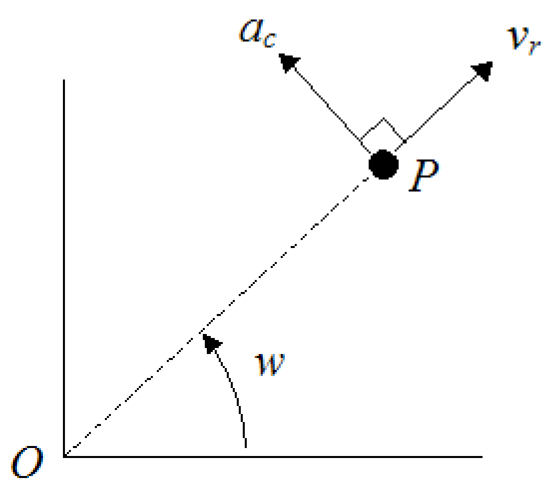
**Bottom panel:** an exploded view in the non-inertial frame. In the non-inertial frame where the car is not accelerating, the force from the axle is balanced by a fictitious backward force âˆ’(m + M)a, a portion âˆ’Ma applied to the car, and âˆ’ma to the passenger. The car is subject to the fictitious force âˆ’Ma and the force (m + M)a from the axle. The difference between these forces ma is applied to the seat, which exerts a reaction âˆ'ma upon the car, so zero net force is applied to the car. The seat (assumed massless) transmits the force ma to the passenger, who is subject also to the fictitious force âˆ'ma, resulting in zero net force on the passenger. The passenger exerts a reaction force âˆ'ma upon the seat, which is therefore compressed. In all frames the compression of the seat is the same, and the force delivered by the axle is the same.

# Coriolis Force

To explain the Coriolis force it is first necessary to explain Coriolis acceleration.

When an object simultaneously rotates about a point and moves relative to that point, an acceleration results from this. This acceleration is called Coriolis acceleration.

To illustrate this acceleration, consider a particle *P* rotating in a plane about point *O* with a constant angular velocity $w$, and moving radially outwards with a velocity $v_r$. The Coriolis acceleration is denoted by $a_c$. It acts in the circumferential direction (perpendicular to $v_r$).



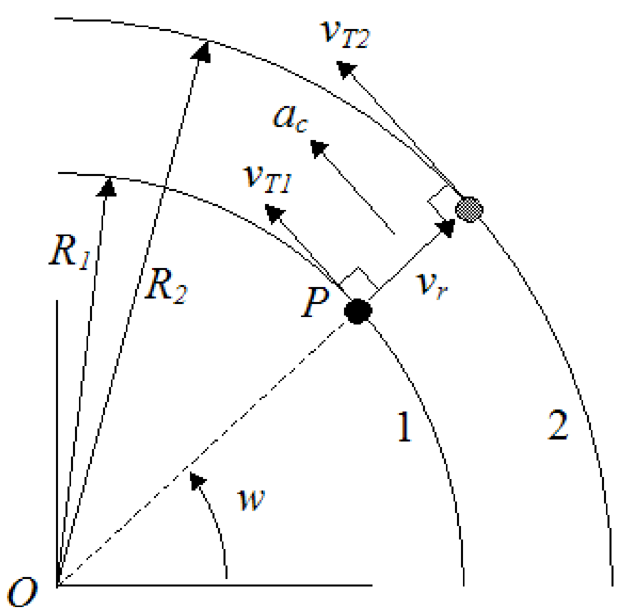

$$a_c = 2w \cdot v_r$$

By applying Newton's Second Law in the direction of $a_c$, we can determine the force acting on the particle $P$ due to this acceleration. Call this force $F_c$.

$$F_c = ma_c = 2m(w \cdot v_r) \quad (1)$$

where $m$ is the mass of the particle.

To gain an intuitive understanding of the acceleration $a_c$, think about what happens as the particle $P$ moves radially outwards. It traces a circle of progressively larger radius. Given that the angular velocity $w$ is constant, and the velocity of the particle tangent to the circle is equal to $wR$ (where $R$ is the radius of the circle), the tangential (circumferential) velocity must then increase as a result. The figure below shows the tangential velocity $v_T$ of the particle $P$ at two consecutive instants.




At instant 1:

$$v_{T1} = wR_1$$

At instant 2:

$$v_{T2} = wR_2$$

Since $R_2 > R_1$,

$$v_{T2} > v_{T1}$$

Therefore there is an acceleration $a_c$ (in the direction shown) due to this tangential increase in velocity.

Thus, the particle $P$ must have a restraining force $F_c$ acting on it in the direction of $a_c$, in order to maintain its radially outward motion.

Note that this restraining force $F_c$ is not necessarily the total force acting on the particle $P$. It is only the part of the force that is due to the combined effect of the particle simultaneously rotating about point $O$ and moving radially outwards from point $O$.

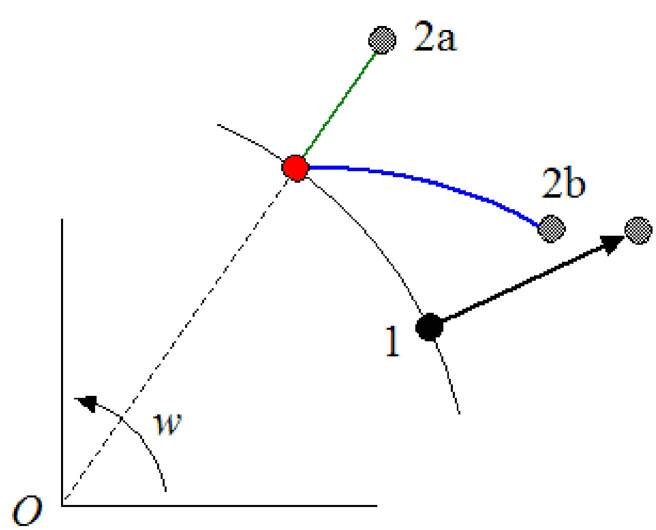
Now, let's define a reference frame that has origin at point $O$ and is rotating at angular velocity $w$. If the restraining force $F_c$ were to vanish (or at least decrease), the particle P would travel a curved path relative to this (rotating) reference frame (as will be explained). The apparent force causing the particle to curve is called the Coriolis force. Note however, that the Coriolis force is a fictitious force, and not a real force, since it is based on motion relative to a non-inertial reference frame, that is rotating. Nevertheless, it can be an informative way of looking at a problem, as will be shown.

For example, let's say we have a merry-go-round rotating with a constant angular velocity. On the merry-go-round there is a ball rolling outwards from the center with a radial velocity $v_r$ (this is the velocity of the ball relative to the merry-go-round). This results in the ball moving in a straight line relative to the merry-go-round (in the direction of $v_r$). The force that maintains this straight-line motion is the restraining force $F_c$ (calculated before).

Now, let's say there is no restraining force ($F_c = 0$). This would result in the ball not being able to increase its tangential speed (since $a_c = 0$ by equation (1)). Consequently, the ball must "fall back", which will result in the ball moving along a curved path relative to the merry-go-round. Thus, from the point of view of an observer sitting on the merry-go-round, the ball would appear to be "pushed" by an imaginary force (called the Coriolis force) causing it to curve.

The figure below shows the motion of the ball relative to the merry-go-round, as the merry-go-round rotates between positions 1 and 2. The figure illustrates two cases: The first case is where the restraining force $F_c$ is present – this is illustrated with the straight (green) line. The second case is where $F_c = 0$ – this is illustrated with the blue curve.

The black arrow shows the motion of the ball relative to an **inertial reference frame** (ground) for the case where there are no external forces acting on the ball in the plane of the merry-go-round.



The red dot is a reference point marked on the merry-go-round. It is defined as coincident with the starting position of the ball, at position 1

In the above figure:

The starting position of the ball is denoted by '1'

In the presence of a restraining force $F_c$ , the final position of the ball is denoted by '2a'

If the restraining force $F_c = 0$ , the final position of the ball is denoted by '2b'

As mentioned already, the green line represents the motion of the ball relative to the merry-go-round for the case where $F_c$ is present. The blue curve represents the motion of the ball relative to the merry-go-round for the case where $F_c = 0$ (but note that even if $F_c = 0$, there may still be other forces acting on the ball in the plane of the merry-go-round).

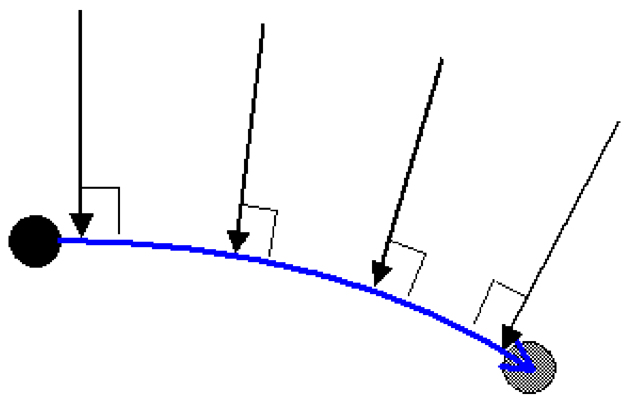
For the situation where $F_c$ is not equal to zero but is less than the restraining force in the first case, the path traveled by the ball (relative to the merry-go-round) would follow a curved path that lies somewhere in between the green line and the blue curve.

From the point of view of an observer sitting on the merry-go-round (and moving with it), the Coriolis force is the apparent force that appears to be acting on the ball, causing it to veer to the right.

More specifically, the Coriolis force acts in a direction that is perpendicular to both the angular velocity vector and the relative (linear) velocity vector.

Referring to the figure above this would mean that the (fictitious) Coriolis force acts perpendicular to the blue curve (where $F_c = 0$), since the angular velocity vector of the merry-go-round is pointing out of the page, and the relative (linear) velocity vector of the ball relative to the merry-go-round is tangent to the blue curve.

The Coriolis force acting on the ball is indicated by the black arrows, shown below.



The Coriolis force explains why the ball tends to curve in on itself, relative to an observer sitting on the merry-go-round (and moving with it).

The black arrow shown in the previous figure (with starting position '1') represents the path traveled by the ball relative to ground (an **inertial reference frame**), for the situation where no external forces are acting on the ball in the plane of the merry-go-round (i.e. there is no friction with the merry-go-round surface). In this situation, the ball would travel a straight line (relative to ground) indicated by the black arrow (due to Newton's First Law). However, relative to the merry-go-round, the ball would follow a curved path, similar to the blue curve.

# Motion in a Central Force Field:-

We now study the properties of a particle of (constant) mass $m$ moving in a particular type of force field, a *central force field*. Central forces are very important in physics and engineering. For example, the gravitional force of attraction between two point masses is a central force. The Coulomb force of attraction and repulsion between charged particles is a central force. Because of their importance they deserve special consideration. We begin by giving a precise definition of *central force, or central force field.*

**Central Forces: The Definition.** Suppose that a force acting on a particle of mass $m$ has the properties that:

- the force is always directed from $m$ toward, or away, from a fixed point $O$,
- the magnitude of the force only depends on the distance $r$ from $O$.

Forces having these properties are called *central forces*. The particle is said to move in a *central force field*. The point $O$ is referred to as the *center of force*.

Mathematically, $\mathbf{F}$ is a central force if and only if:

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\frac{\mathbf{r}}{r}, \tag{1}$$

where $\mathbf{r}_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of $\mathbf{r}$.

If $f(r) < 0$ the force is said to be *attractive towards O*. If $f(r) > 0$ the force is said to be *repulsive from O*. We give a geometrical illustration in Fig. 1.

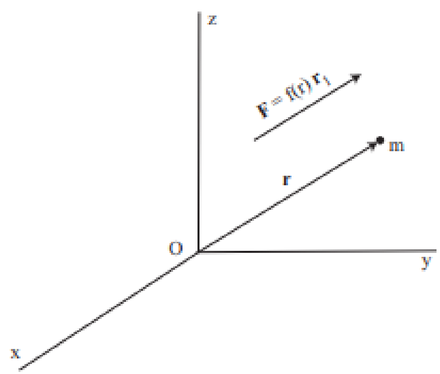



Figure 1: Geometrical illustration of a central force.

**Properties of a Particle Moving under the Influence of a Central Force.** If a particle moves in a central force field then the following properties hold:

1. The path of the particle must be a *plane curve*, i.e., it must lie in a plane.
2. The angular momentum of the particle is conserved, i.e., it is constant in time.
3. The particle moves in such a way that the position vector (from the point $O$) sweeps out equal areas in equal times. In other words, the time rate of change in area is constant. This is referred to as the *Law of Areas*. We will describe this in more detail, and prove it, shortly.

**Equations of Motion for a Particle in a Central Force Field.** Now we will derive the basic equations of motion for a particle moving in a central force field.

From Property 1 above, the motion of the particle must occur in a plane, which we take as the $xy$ plane, and the center of force is taken as the origin. In Fig. 2 we show the $xy$ plane, as well as the polar coordinate system in the plane.

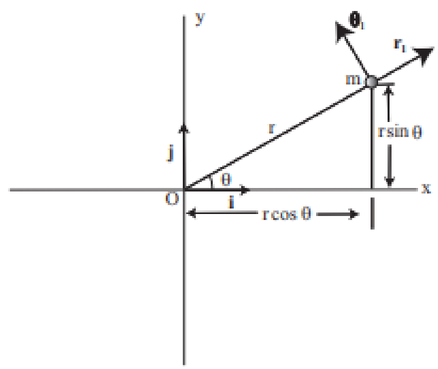



Figure 2: Polar coordinate system associated with a particle moving in the $xy$ plane.

Since the vectorial nature of the central force is expressed in terms of a radial vector from the origin it is most natural (though not required!) to write the equations of motion in polar coordinates. In earlier lectures we derived the expression for the acceleration of a particle in polar coordinates:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\theta}_1. \tag{2}$$

Then, using Newton's second law, and the mathematical form for the central force given in (1), we have:

$$m(\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\theta}_1 = f(r)\mathbf{r}_1, \tag{3}$$

# Gauss' Law For Gravitation

- We can define the gravitational field flux through a surface as:

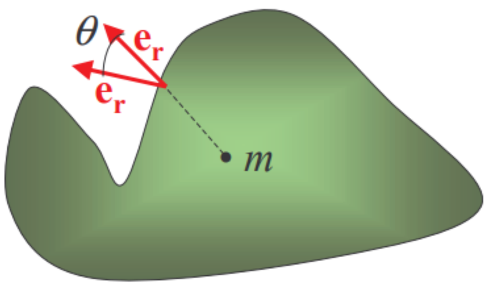
$$\Phi_m = \int_S \mathbf{g} \cdot \mathbf{n} da$$

where **n** is a unit vector perpendicular to the surface at each point

- For the case of a closed surface around a point mass $m$, we have:

$$\Phi_m = \int_S \mathbf{g} \cdot \mathbf{n} da = \int_S \left( \frac{-Gm}{r^2} \right) \mathbf{e_r} \cdot \mathbf{n} da$$

$$= -Gm \int_S \frac{\cos\theta}{r^2} da$$




- If we take the special case where the surface is a sphere, the integral is easy:

$$\Phi_m = -Gm \int_S \frac{1}{r^2} da = \frac{-Gm}{r^2} \int_S da$$

$$= -4\pi Gm$$

- Physically, we can interpret the flux as the "number" of gravitational field lines passing through the surface
  - But since the lines start at a point, and extend an infinite distance away, the flux can't depend on the shape of the surface that encloses our point

- Thus we have the *general* result, for any closed surface around a point mass,

$$\Phi_m = -4\pi G m$$

Note that it also doesn't matter *where* the mass is

- This can be easily extended to the case where there are $N$ point masses inside the surface. Since the field is linear,

$$\Phi_m = -4\pi G \sum_{i=1}^{N} m_i$$

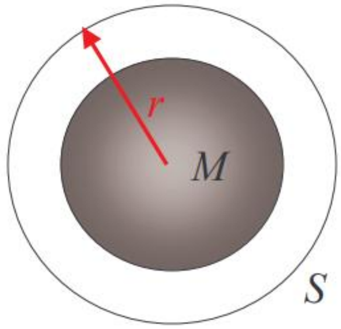
- For a continuous distribution of matter, it's:

$$\Phi_m = -4\pi G \int_V \rho(\mathbf{r})\, dv$$

- For problems involving symmetric distributions of matter, Gauss' Law is a useful shortcut to finding the field

## Example: Field Due to a Sphere

- Assume we have a spherically symmetric mass distribution (with the density varying as a function of the distance from the sphere's center)
- We want to find the field at any point external to the sphere
  – The symmetry of the problem makes is a good candidate for Gauss' Law
  – We should choose a surface that reflects this symmetry:




$$\int_S \mathbf{g}\cdot\mathbf{n}da = -4\pi GM$$

From symmetry, we know that:
$\mathbf{g} = g_r(r)\mathbf{e_r}$ and $\mathbf{n} = \mathbf{e_r}$

- Therefore, the integral becomes much simpler:

$$\int_S \mathbf{g}\cdot\mathbf{n}da = \int_S g(r)da = 4\pi r^2 g(r)$$

$$4\pi r^2 g(r) = -4\pi GM$$

$$g(r) = \frac{-GM}{r^2}$$

- From this, we see that the field due to a sphere is exactly the same as if all the mass were concentrated at the center of the sphere
  - This is not true for other shapes
- This result was very important for Newton, since it justified his treatment of the Earth as a point mass when calculating the motion of the moon

## Poisson's Equation

- We can look at Gauss' Law another way to find another important property of the gravitational potential:

$$\int_S \mathbf{g}\cdot\mathbf{n}da = \int_V \nabla\cdot\mathbf{g}dv$$

This is just the divergence theorem

$$= \int_V \left(\nabla\cdot\nabla\Phi\right)dv = \int_V \nabla^2\Phi dv$$

$$= -4\pi G\int_V \rho\left(\mathbf{r}'\right)dv$$

- For the last relation to hold for an arbitrary volume $V$, the integrands must be the same everywhere:

$$\nabla^2\Phi = -4\pi GM\rho\left(\mathbf{r}'\right)$$

## Laplace's Equation

- In the special case where there is no material in a region of space, the potential in that region satisfies Laplace's Equation:

$$\nabla^2\Phi = 0$$

- Intuitively, this is nothing more than the statement that field lines can't start (or end) in a region where there is no mass
- Mathematically, this gives us a way to determine the potential in any mass-free region

- As long as the bounday conditions (the value of the potential at the edges of the region) are specified

- In practice, this equation is more useful in calculating *electric* potentials than for gravitational potentials…

### Kinetic energy of the system of particles

- Let there are n number of particles in a n particle system and these particles possess some motion. The motion of the i'th particle of this system would depend on the external force $\mathbf{F}_i$ acting on it. Let at any time if the velocity of i'th particle be $\mathbf{v}_i$ then its kinetic energy would be

$$E_{Ki} = \frac{1}{2}mv_i^2$$

$$E_{Ki} = \frac{1}{2}m(\boldsymbol{v}_i.\boldsymbol{v}_i) \qquad (1)$$

- Let $\mathbf{r}_i$ be the position vector of the i'th particle w.r.t. O and $\mathbf{r'}_i$ be the position vector of the centre of mass w.r.t. $\mathbf{r}_i$ ,as shown below in the figure , then

$\mathbf{r}_i=\mathbf{r'}_i+\mathbf{R}_{cm}$ (2)

where $\mathbf{R}_{cm}$ is the position vector of centre of mass of the system w.r.t. O.

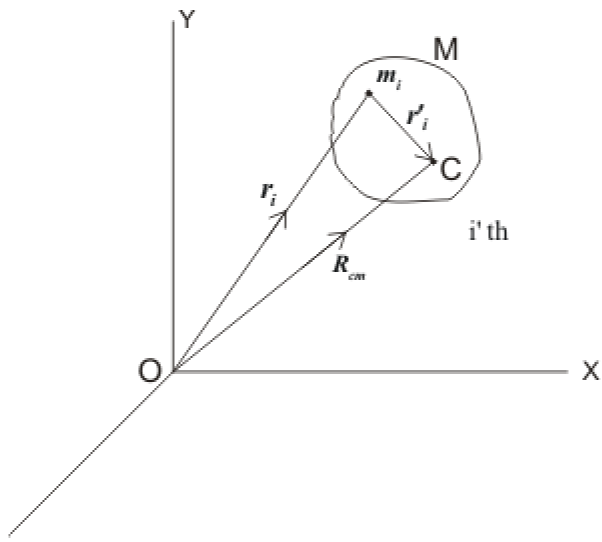



Figure 4. $r'_i$ Is the position vector of center of mass w.r.t. $r_i$

- Differentiating equation 2 we get

$$\frac{d\boldsymbol{r}_i}{dt} = \frac{d\boldsymbol{r}'_i}{dt} + \frac{d\boldsymbol{R}_{cm}}{dt}$$

or,

$$\boldsymbol{v}_i = \boldsymbol{v}_i' + \boldsymbol{V}_{cm} \qquad (3)$$

where $\mathbf{v}_i$ is the velocity of i'th particle w.r.t. centre of mass and $\mathbf{V}_{cm}$ is the velocity of centre of mass of system of particle. Putting equation 3 in 1 we get,

$$E_{Ki} = \frac{1}{2} m_i \left[ (\boldsymbol{v}_i' + \boldsymbol{V}_{cm}).(\boldsymbol{v}_i' + \boldsymbol{V}_{cm}) \right] = \frac{1}{2} m_i \left[ (\boldsymbol{v}_i^{2\prime} + 2\boldsymbol{v}'_i . \boldsymbol{V}_{cm} + \boldsymbol{V}_{cm}^2) \right]$$

$$E_{Ki} = \frac{1}{2} m_i \boldsymbol{v}_i'^2 + m_i \boldsymbol{v}'_i . \boldsymbol{V}_{cm} + \frac{1}{2} m_i \boldsymbol{V}_{cm}^2 \qquad (4)$$

- Sum of Kinetic energy of all the particles can be obtained from equation 4

$$E_K = \sum_{i=1}^{n} E_{Ki} = \sum_{i=1}^{n} \left[ \frac{1}{2} m_i \boldsymbol{v}_i'^2 + m_i \boldsymbol{v}'_i . \boldsymbol{V}_{cm} + \frac{1}{2} m_i \boldsymbol{V}_{cm}^2 \right]$$

$$E_K = \sum_{i=1}^{n} \frac{1}{2} m_i \boldsymbol{v}_i'^2 + \sum_{i=1}^{n} m_i \boldsymbol{v}'_i . \boldsymbol{V}_{cm} + \sum_{i=1}^{n} \frac{1}{2} m_i \boldsymbol{V}_{cm}^2$$

$$E_K = \frac{1}{2} \boldsymbol{V}_{cm}^2 \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} \frac{1}{2} m_i \boldsymbol{v}_i'^2 + \boldsymbol{V}_{cm} \sum_{i=1}^{n} m_i \boldsymbol{v}'_i$$

$$E_K = \frac{1}{2} \boldsymbol{V}_{cm}^2 M + \sum_{i=1}^{n} \frac{1}{2} m_i \boldsymbol{v}_i'^2 + \boldsymbol{V}_{cm} \frac{d}{dt} \sum_{i=1}^{n} m_i \boldsymbol{r}'_i \qquad (5)$$

- Now last term in above equation which is

$$\sum_{i=1}^{n} m_i \boldsymbol{r}'_i = 0$$

would vanish as it defines the null vector because

$$\sum_{i=1}^{n} m_i \boldsymbol{r}'_i = \sum_{i=1}^{n} m_i (\boldsymbol{r}_i - \boldsymbol{R}_{cm}) = M\boldsymbol{R}_{cm} - M\boldsymbol{R}_{cm} = 0$$

- Therefore kinetic energy of the system of particles is,

$$E_K = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \sum_{i=1}^{n} m_i v_i'^2 = E_{Kcm} + E'_K \qquad (6)$$

where,

$$E_{Kcm} = \frac{1}{2} V_{cm}^2 M$$

is the kinetic energy obtained as if all the mass were concentrated at the centre of mass and

$$E'_K = \sum_{i=1}^{n} \frac{1}{2} m_i v_i'^2$$

is the kinetic energy of the system of particle w.r.t. the centre of mass.

- Hence it is clear from equation 6 that kinetic energy of the system of particles consists of two parts: the kinetic energy obtained as if all the mass were concentrated at the centre of mass plus the kinetic energy of motion about the centre of mass.
- If there were no external force acting on the particle system then the velocity of the centre of mass of the system will remain constant and Kinetic Energy of the system would also remain constant.

### Two particle system and reduced mass

- Two body problems with central forces can always be reduced to the form of one body problems.
- Consider a system made up of two particles. For an observer in any inertial frame of refrence relative motion of these two particles can be represented by the motion of a fictitious particle.
- The mass of this fictetious particle is known as the reduced mass of two particle system.
- Consider a system of two particles of mass $m_1$ and $m_2$ respectively. Let O be the origin of any inertial frame of refrance and $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of these particles at any time t w.r.t. origin O as shown bellow in the figure.

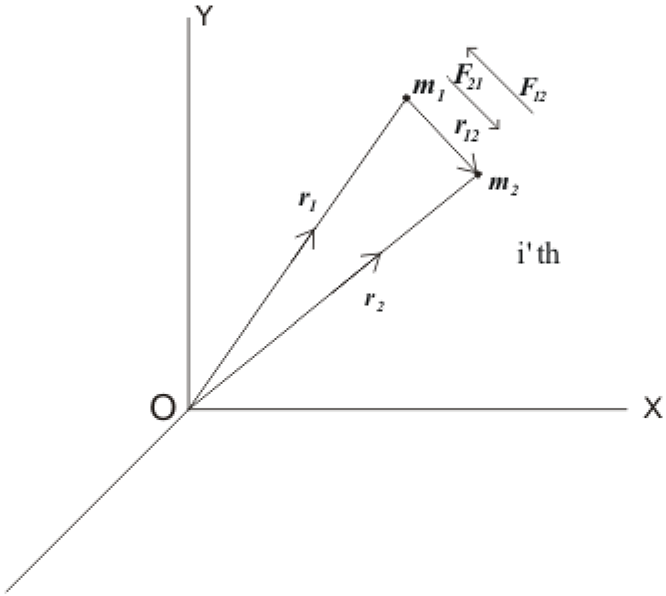



Figure 5. Two particle system

- If no external force is acting on the system then the force acting on the system would be equal to mutual interaction between two particles. Let the force acting on $m_1$ due to $m_2$ be $\mathbf{F}_{21}$ and force acting on $m_2$ due to $m_1$ be $\mathbf{F}_{12}$ then equation of motion for particles $m_1$ and $m_2$ would be

$$\boldsymbol{F}_{21} = m_1 \frac{d^2 \boldsymbol{r}_1}{dt^2} \qquad (1)$$

and,

$$\boldsymbol{F}_{12} = m_2 \frac{d^2 \boldsymbol{r}_2}{dt^2} \qquad (2)$$

from 1 and 2

$$\frac{d^2 \boldsymbol{r}_1}{dt^2} = \frac{\boldsymbol{F}_{21}}{m_1} \qquad (3)$$

and

$$\frac{d^2 \boldsymbol{r}_2}{dt^2} = \frac{\boldsymbol{F}_{12}}{m_2} \qquad (4)$$

From the figure

$$\boldsymbol{r}_{12} = \boldsymbol{r}_2 - \boldsymbol{r}_1 \qquad (5)$$

so,

$$\frac{d^2 \boldsymbol{r}_{12}}{dt^2} = \frac{d^2 \boldsymbol{r}_2}{dt^2} - \frac{d^2 \boldsymbol{r}_1}{dt^2} \qquad (6)$$

putting 3 and 4 in 6 we get

$$\frac{d^2 \boldsymbol{r}_{12}}{dt^2} = \frac{\boldsymbol{F}_{12}}{m_2} - \frac{\boldsymbol{F}_{21}}{m_1} \qquad (7)$$

but from Newton's first law of motion we have
$\mathbf{F}_{21} = -\mathbf{F}_{12}$
then from equation 7 we have

$$\frac{d^2\boldsymbol{r}_{12}}{dt^2} = \frac{\boldsymbol{F}_{12}}{m_2} + \frac{\boldsymbol{F}_{12}}{m_1} = \boldsymbol{F}_{12}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$\text{or, } \frac{d^2\boldsymbol{r}_{12}}{dt^2} = \boldsymbol{F}_{12}\left(\frac{m_1 + m_2}{m_1 m_2}\right)$$

$$\boldsymbol{F}_{12} = \left(\frac{m_1 m_2}{m_1 + m_2}\right)\frac{d^2\boldsymbol{r}_{12}}{dt^2}$$

$$\text{or, } \boldsymbol{F}_{12} = \mu \frac{d^2\boldsymbol{r}_{12}}{dt^2} \qquad (8)$$

$$\text{where, } \mu = \left(\frac{m_1 m_2}{m_1 + m_2}\right)$$

is known as reduced mass of the system.

- This equation 8 represents a one body problem , because it is similar to the equation of motion of single particle of mass μ at a vector distance $\mathbf{r}_{12}$ from one of thr two particles, considered as the fixed centre of force.
- Thus original problem involving two particle system has now been reduced to that of one particle system which is easier to solve then original two body problem.

**Case 1.** $m_1 << m_2$

- If the mass of any one particle in two particle system is very very less in comparison to other particle like in earth-satellite system then reduced mass of the system would be

$$\mu = \left(\frac{m_1 m_2}{m_1 + m_2}\right) = \left(\frac{m_1}{1 + (m_1 / m_2)}\right)$$

$$\text{or, } \mu \approx \left(1 - \frac{m_1}{m_2}\right) m_1 \qquad \text{(using binomial theorem)}$$

$$\text{or, } \mu \approx m_1$$

- So the reduced mass of the two particle system would be equal to the particle having lesser mass.

**Case 2.** $m_1 = m_2 = m$

- If the masses o8 the particles of a two particle system are same then

$$\mu = \left(\frac{m_1 m_2}{m_1 + m_2}\right) = \frac{m^2}{2m} = \frac{m}{2}$$

- Hence reduced mass of the system would be equal to the one half of the mass of a single particle.

### (9) Linear momentum and principle of conservation of linear momentum

- Product of mass and velocity of any particle is defined as the linear momentum of the particle. It is a vector quantity and its direction is same as the direction of velocity of the particle.
- Linear momentum is represented by **p**. If m is the mass of the particle moving with velocity **v** then linear momentum of the particle would be
  **p**=m**v**
  like **v** , **p** also depends on the frame of refrance of the observer.
- If in a many particle system $m_1$ , $m_2$ , $m_3$ , . . . . . . . . . . . . . . , $m_n$ are the masses and $\mathbf{v}_1$ , $\mathbf{v}_2$ , $\mathbf{v}_3$, . . . . . . . . . . . . . ., $\mathbf{v}_n$ are the velocities of the respective particles then total linear momentum of the system would be

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots\ldots\ldots. + \mathbf{p}_n \qquad (2)$$

$$\mathbf{p} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + \ldots\ldots\ldots + m_n\mathbf{v}_n$$

$$\mathbf{p} = M\mathbf{V}_{cm}$$

  where M is the total mass of the system and $\mathbf{V}_{cm}$ is the velocity of centre of mass of the system
- Hence from equation 2 we came to know that total linear momentum of a many particle system is equal to the product of the total mass of the system and velocity of centre of mass of the system.

- Differentiating equation 2 w.r.t. t we get

$$\frac{d\boldsymbol{p}}{dt} = M\frac{d\boldsymbol{V}_{cm}}{dt} = M\boldsymbol{a}_{cm} \qquad (3)$$

  but , $M\mathbf{a}_{cm}$=$\mathbf{F}_{ext}$ which is the external force acting on the system. Therefore,

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}_{ext} \qquad (4)$$

  like this the rate of change of momentum of a many particle system comes out to be equal to the resultant external force acting on the particle.

- If external force acting on the system is zero then,

$$\frac{d\boldsymbol{p}}{dt} = 0$$

$$\text{or, } \boldsymbol{p} = \text{constant}$$

$$\text{or, } \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 + \ldots\ldots + \boldsymbol{p}_1 = \text{const.}$$

$$\left.\begin{aligned} M\boldsymbol{V}_{cm} &= \boldsymbol{p} = \text{const.} \\ \text{or, } \boldsymbol{V}_{cm} &= \text{const.} \end{aligned}\right\} \quad (5)$$

- Hence we conclude that when resultant external force acting on any particle is zero then total linear momentum of the system remains constant. This is known as law of conservation of linear momentum.
- Above equation 5 is equivalent to following scalar quantities

$$\left.\begin{aligned} p_{x1} + p_{x2} + p_{x3} + \ldots\ldots\ldots + p_{xn} &= \text{const.} \\ p_{y1} + p_{y2} + p_{y3} + \ldots\ldots\ldots + p_{yn} &= \text{const.} \\ p_{z1} + p_{z2} + p_{z3} + \ldots\ldots\ldots + p_{zn} &= \text{const.} \end{aligned}\right\} \quad (6)$$

  Equation 6 shows the total linear momentum of the system in terms of x , y and z co-ordinates and also shows that they remain constant or conserved in absence of any externally applied force.
- The law of conservation of linear momentum is the fundamental and exact law of nature. No violation of it has ever been found. This law has been established on the basis of Newton's law but this law holds true in the situations where Newtonian mechanics fails.

## Centre of mass frame of refrance

- If we attach an inertial frame of refrance with the centre of mass of many particle system then centre of mass in that frame of refrance would be at rest or, $\mathbf{V}_{cm}$=0 , and such type of refrance frames are known as centre of mass frame of refrance.
- Total linear momentum of a many particle system is zero in centre of mass frame of refrance i.e., $\mathbf{p}_{cm}=M\mathbf{V}_{cm}=0$ since $\mathbf{V}_{cm}=0$.
- Therefore C-refrance frames are also known as zero momentum refrance frames.
- Since in absence of any external force the centre of mass of any system moves with constant velocity in inertial frame of refrance therefore for a many particle system C-rame of refrance is an inertial frame of refrence.
- Refrance frames connected to laboratory are known as L-frame of refrance or lebiratory frame of refrance.

## Collisions

- Collision between two particles is defined as mutual interaction between particles for a short interval of time as a result of which energy and momentum of particle changes.
- Collision between two billiard balls or between two automobiles on road are few examples of collisions from our everyday life. Even gas atoms and molecules at room temperature keep on colliding against each other.
- For the collision to take place , physical contact is not necessary. In cas of Rutherford alpha scattering experiment , the alpha particles are scattered due to electrostatic interaction between the alpha particles and the nucleus from a distance i.e., no physical contact occurs between the alpha particles and the nucleus.
- Thus , in physics collision is said to have occured , if two particles physically collide with each other or even when the path of motion of one particle is affected by other.
- In the collision of two particles law of conservation of momentum always holds true but in some collisions Kinetic energy is not always conserved.
- Hence collisions are of two types on the basis of conservation of energy.
  (i) Perfectly elastic collision
- Those collisions in which both momentum and kinetic energy of system are conserved are called elastic collisions for example elastic collision occurs between the molecules of a gas. This type of collision mostly takes place between the atoms, electrons and protons.
- **Characterstics of elastic collision**
  (a) Total momentum is conserved.
  (b) Total energy is conserved.
  (c) Total kinetic energy is conserved.
  (d) Total mechanical energy is not converted into any other form of energy.
  (e) Forces involved during interaction are conservative in nature.
- Consider two particles whose masses are $m_1$ and $m_2$ respectively and they collide each other with velocity $\mathbf{u}_1$ and $\mathbf{u}_2$ and after collision their velocities
  become $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.
- If collision between these two particles is elastic one then from law of conservation of momentum we have
  $m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$
  and from the law of conservation of energy we have
  $$\frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2 = \frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2$$
  (ii) Perfectly inelastic collision
- Those collisions in which momentum of system is conserved but kinetic energy of the system is not conserved are known as inelastic collision.
- Here in inelastic collision two bodies stick to each other after collision as a bullet hit its target and remain embedded in the target.
- In this case some of the kinetic energy is converted into heat or is used up in in doing work in deforming bodies for example when two cars collide their metal parts are bet out of shape.
- **Characterstics of inelastic collision**
  (a) Total momentum is conserved.
  (b) Total energy is conserved.

(c) Total kinetic energy is not conserved.
(d) A part or whole of whole mechanical energy may be converted into other forms of energy.
(e) Some or all forces involved during interaction are non-conservative in nature.

- Consider two particles whose masses are $m_1$ and $m_2$ respectively and they collide each other with velocity $\mathbf{u}_1$ and $\mathbf{u}_2$ respectively.
- If the collision between these two particles is inelastic then these two particles would stick to each other and after collision they move with velocity $\mathbf{v}$ then from law of conservation of momentum we have

$$m_1\boldsymbol{u}_1 + m_2\boldsymbol{u}_2 = (m_1 + m_2)\boldsymbol{v}$$

$$\boldsymbol{v} = \frac{m_1\boldsymbol{u}_1 + m_2\boldsymbol{u}_2}{(m_1 + m_2)}$$

- Kinetic energy of particles before collisions is

$$K.E._i = \frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2$$

and kinetic energy of particles after collisions is

$$K.E._f = \frac{1}{2}(m_1 + m_2)\mathbf{v}^2$$

by law of conserevation of energy

$$\frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2 = \frac{1}{2}(m_1 + m_2)\mathbf{v}^2 + Q$$

where Q is the loss in kinetic energy of particles during collision.

## Head on elastic collision of two particles in L-frame of refrance

- Consider two particles whose masses are $m_1$ and $m_2$ respectively and they collide each other with velocity $\mathbf{u}_1$ and $\mathbf{u}_2$ and after collision their velocities become $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.
- Collision between these two particles is head on elastic collision. From law of conservation of momentum we have

$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ (1)

and from law of conservation of kinetic energy for elastic collision we have

$$\frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2 = \frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2 \qquad (2)$$

rearranging equation 1 and 2 we get

$m_1(\mathbf{u}_1-\mathbf{v}_1)= m_2(\mathbf{v}_2-\mathbf{u}_2)$ (3)

and

$$m_1(\mathbf{u}_1^2 - \mathbf{v}_1^2) = m_2(\mathbf{v}_2^2 - \mathbf{u}_2^2) \qquad (4)$$

dividing equation 4 by 3 we get
$\mathbf{u}_1 + \mathbf{v}_1 = \mathbf{u}_2 + \mathbf{v}_2$
$\mathbf{u}_2 - \mathbf{u}_1 = -(\mathbf{v}_2 - \mathbf{v}_1)$ (5)
where ($\mathbf{u}_2 - \mathbf{u}_1$) is the relative velocity of second particle w.r.t. first particle before collision and ($\mathbf{v}_2 - \mathbf{v}_1$) is the relative velocity of second particle w.r.t. first after collision.

- From equation 5 we come to know taht in a perfectly elastic collision the magnitude of relative velocity remain unchanged but its direction is reversed. With the help of above equations we can find the values of $\mathbf{v}_2$ and $\mathbf{v}_1$ , so from equation 5
  $\mathbf{v}_1 = \mathbf{v}_2 - \mathbf{u}_1 + \mathbf{u}_2$ (6)
  $\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{u}_1 - \mathbf{u}_2$ (7)
  Now putting the value of $\mathbf{v}_1$ from equation 6 in equation 3 we get
  $m_1(\mathbf{u}_1 - \mathbf{v}_2 + \mathbf{u}_1 - \mathbf{u}_2) = m_2(\mathbf{v}_2 - \mathbf{u}_2)$
  On solving the above equation we get value of $\mathbf{v}_2$ as

$$\mathbf{v}_2 = \left(\frac{2m_1}{m_1 + m_2}\right)\mathbf{u}_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)\mathbf{u}_2 \qquad (8)$$

- Similarly putting the value of $\mathbf{v}_2$ from equation 7 in equation 3 we get

$$\mathbf{v}_1 = \left(\frac{2m_2}{m_1 + m_2}\right)\mathbf{u}_2 + \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\mathbf{u}_1 \qquad (9)$$

- Total kinetic energy of particles before collision is

$$KE_i = \frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2$$

  and total K.E. of particles after collision is

$$KE_f = \frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2$$

- Ratio of initial and final K.E. is

$$\frac{KE_i}{KE_f} = \frac{\frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2}{\frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2} = 1$$

- **Special cases**
  **Case I:** When the mass of both the particles are equal i.e., $m_1 = m_2$ then from equation 8 and 9 , $\mathbf{v}_2=\mathbf{u}_1$ and $\mathbf{v}_1=\mathbf{u}_2$. Thus if two bodies of equal masses suffer head on elastic collision then the particles will exchange their velocities. Exchange of momentum between two particles suffering head on elastic collision is maximum when mass of both the particles is same.

  **Case II:** when the target particle is at rest i,e $u_2=0$
  From equation (8) and (9)

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 \qquad \text{---(10)}$$

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 \qquad \text{---(11)}$$

Hence some part of the KE which is transformed into second particle would be

$$\frac{\frac{1}{2}m_2 v_2^2}{\frac{1}{2}m_1 u_1^2} = \frac{\frac{1}{2}m_2\left(\frac{2m_1 u_1}{m_1 + m_2}\right)^2}{\frac{1}{2}m_1 u_1^2}$$

$$= \frac{4m_1 m_2}{(m_1 + m_2)^2} = \frac{4\frac{m_2}{m_1}}{(1 + \frac{m_2}{m_1})^2} \qquad \text{--(12)}$$

when $m_1$=$m_2$,then in this condition $v_0$=0 and $v_2$=$u_1$ and part of the KE transferred would be
=1

Therefore after collisom first particle moving with initial velocity $u_1$ would come to rest and the second particle which was at rest would start moving with the velocity of first particle.Hence in this case when $m_1$=$m_2$ transfer of energy is 100%.if $m_1 > m_2$ or $m_1 < m_2$ ,then energy transformation is not 100%

**Case III:**
if $m_2$ >>>> $m_1$ and $u_2$=0 then from equation (10) and (11)
$v_1 \cong -u_1$ and $v_2$=0 (13)
For example when a ball thrown upwards collide with earth

**Case IV:**
if $m_1$ >>>> $m_2$ and $u_2$=0 then from equation (10) and (11)
$v_1 \cong u_1$ and $v_2$=$2u_1$ (14)

Therefore when a heavy particle collide with a very light particle at rest ,then the heavy particle keeps on moving with the same velocity and the light particle come in motion with a velocity double that of heavy particle

## Head on collison of two particles in C frame of refrence

- Consider two particles of mass $m_1$ and $m_2$ having position vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ respectively And position vector of the CEnter of mass of the system would be $\mathbf{R_{cm}}$ then

$$\mathbf{R}_{cm} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \qquad \text{--(15)}$$

Velocity of the center of mass would be

$$\boldsymbol{v}_{cm} = \frac{d\boldsymbol{R}_{cm}}{dt} = \frac{m_1\frac{d\boldsymbol{r}_1}{dt} + m_2\frac{d\boldsymbol{r}_2}{dt}}{m_1 + m_2} = \frac{m_1\boldsymbol{u}_1 + m_2\boldsymbol{u}_2}{m_1 + m_2} \qquad \text{--(16)}$$

Intial velocity of the $m_1$ w.r.t center of mass frame of refrence is

$$\boldsymbol{u}_1' = \boldsymbol{u}_1 - \boldsymbol{v}_{cm} = \boldsymbol{u}_1 - \left(\frac{m_1\boldsymbol{u}_1 + m_2\boldsymbol{u}_2}{m_1 + m_2}\right)$$

$$= \frac{m_1\boldsymbol{u}_1 + m_2\boldsymbol{u}_1 - m_1\boldsymbol{u}_1 - m_2\boldsymbol{u}_2}{m_1 + m_2} = \frac{m_2\boldsymbol{u}_1 - m_2\boldsymbol{u}_2}{m_1 + m_2} \qquad \text{--(17)}$$

Similarly Intial velocity of $m_2$ w.r.t center of mass frame of refrence is

$$\boldsymbol{u}_2' = \boldsymbol{u}_2 - \boldsymbol{v}_{cm} = \boldsymbol{u}_2 - \left(\frac{m_1\boldsymbol{u}_1 + m_2\boldsymbol{u}_2}{m_1 + m_2}\right) = \frac{m_1\boldsymbol{u}_2 - m_1\boldsymbol{u}_1}{m_1 + m_2} \qquad \text{--(18)}$$

- Total linear momentum before collison in absence of external force in C frame of refrence would be
$=m_1\mathbf{u_1}' + m_2\mathbf{u_2}'$
$=0$

So $\mathbf{u_2}' = (m_1/m_2)\mathbf{u_1}'$

- If $\mathbf{v_1}'$ and $\mathbf{v_2}'$ are the velocites of mass $m_1$ and $m_2$ respectively after collision then by law of conservation of linear momentum
$m_1\mathbf{v_1}' + m_2\mathbf{v_2}' = 0$
$\mathbf{v_2}' = (m_1/m_2)\mathbf{v_1}'$

Since the collision is elastic,Kinetic energy will be conserved

$$\frac{1}{2}m_1u_1^{'2}+\frac{1}{2}m_2u_2^{'2}=\frac{1}{2}m_1v_1^{'2}+\frac{1}{2}m_2v_2^{'2}$$

Putting the values of $u_2^{'}$ and $v_2^{'}$ ,we get

$$m_1u_1^{'2}+m_2\left(\frac{m_1u_1^{'}}{m_2}\right)^2=m_1v_1^{'2}+m_2\left(\frac{m_1v_1^{'}}{m_2}\right)^2$$

hence $v_1^{'2}=u_1^{'2}$

From which $|\mathbf{v_1}^{'}|=|\mathbf{u_1}^{'}|$ and $|\mathbf{v_2}^{'}|=|\mathbf{u_2}^{'}|$

hence after collison velocities of particles remain unchanged in center of mass frame of refrence.If the collision is one dimmension then because of the collsion direction of these would be opposite to that of their intial velocites

$$\mathbf{v}_1^{'}=-\mathbf{u}_1^{'}=-(\frac{m_2\mathbf{u}_1-m_2\mathbf{u}_2}{m_1+m_2}) \qquad \text{--(19)}$$

$$\mathbf{v}_2^{'}=-\mathbf{u}_2^{'}=-(\frac{m_1\mathbf{u}_2-m_1\mathbf{u}_1}{m_1+m_2}) \qquad \text{--(20)}$$

# UNIT-3

# Elastic Moduli

In the stress-strain curve given below, the region within the elastic limit (region OA) is of importance to structural and manufacturing sectors since it describes the maximum stress a particular material can take before being permanently deformed. The modulus of elasticity is simply the ratio between stress and strain. Elastic Moduli can be of three types, Young's modulus, Shear modulus, and Bulk modulus. In this article, we will understand elastic moduli in detail.

## Elastic Moduli – Young's Modulus

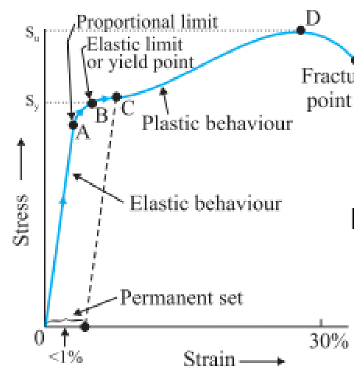



Fig. 1

Many experiments show that for a given material, the magnitude of strain produces is the same regardless of the stress being tensile or compressive. Young's modulus (Y) is the ratio of the tensile/compressive stress (σ) to the longitudinal strain (ε).

$$Y = \frac{\sigma}{\varepsilon} \ldots (1)$$

We already know that, the magnitude of stress = $\frac{F}{A}$ and longitudinal strain = $\frac{\Delta L}{L}$. Substituting these values, we get

$$Y = \frac{\frac{F}{A}}{\frac{\Delta L}{L}}$$

$$\therefore Y = \frac{(F \times L)}{(A \times \Delta L)} \ldots (2)$$

Now, Strain is a dimensionless quantity. Hence, the unit of Young's modulus is N/m2 or Pascal (Pa), the same as that of stress. Let's look at Young's moduli and yield strengths of some materials now:

| Materials | Young's Modulus Y ($10^9$ N/m$^2$) | Elastic Limit ($10^7$ N/m$^2$) | Tensile Strength ($10^7$ N/m$^2$) |
|---|---|---|---|
| Aluminum | 70 | 18 | 20 |
| Copper | 120 | 20 | 40 |
| Wrought Iron | 190 | 17 | 33 |
| Steel | 200 | 30 | 50 |
| Bone | | | |
| Tensile | 16 | – | 12 |
| Compressive | 9 | – | 12 |

From the table, you can observe that Young's moduli for metals are large. This means that metals require a large force to produce a small change in length. Hence, the force required to increase the length of a thin wire of steel is much larger than that required for aluminum or copper. Therefore, steel is more elastic than the other metals in the table.

## Determination of Young's Modulus of the Material of a Wire

The figure below shows an experiment to determine Young's modulus of a material of wire under tension.

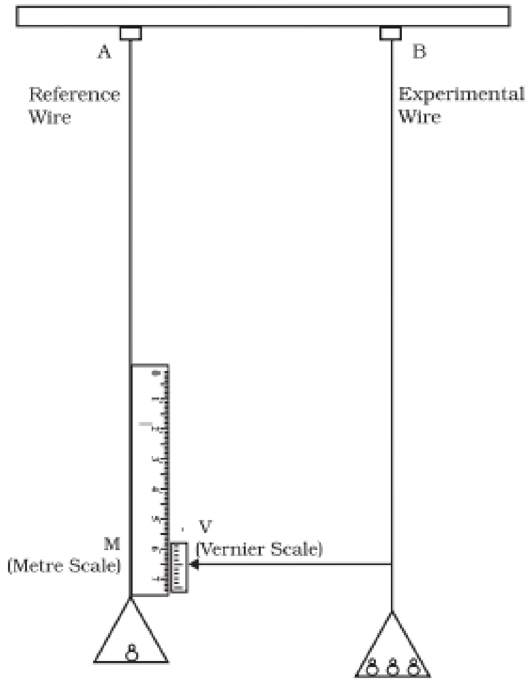



Fig. 2 *An arrangement for the determination of Young's modulus of the material of a wire.*

As can be seen in the diagram above, the setup consists of two long and straight wires having the same length and equal radius. These wires are suspended side-by-side from a fixed rigid support. The reference wire (wire A) has a millimeter main scale (M) and a pan to place weight.

The experimental wire (wire B) also has a pan in which we can place weights. Further, a vernier scale is attached to a pointer at the bottom of wire B and the scale M is fixed to reference wire A. Now, we place a small weight in both the pans to keep the wires straight and note the vernier scale reading.

Next, the wire B is slowly loaded with more weights, bringing it under tensile stress and the vernier reading is noted. The difference between the two readings gives the elongation produced in the wire. The reference wire A is used to compensate for any change in length due to a change in the temperature of the room.

Let r and L be the initial and final length of the wire B, respectively. Therefore, the area of the cross-section of the wire B is = $\pi r^2$. Now, let M be the mass that produces an elongation of $\Delta L$ in wire B. Therefore, the applied force is = Mg, where 'g' is the acceleration due to gravity. Hence, using equations (1) and (2), Young's modulus of the material of wire B is:

$$Y = \frac{\sigma}{\varepsilon} = \frac{Mg}{\pi r^2} \cdot \frac{L}{\Delta L}$$

$$\Rightarrow Y = \frac{(Mg \times L)}{(\pi r^2 \times \Delta L)} \quad \ldots (3)$$

## Elastic Moduli – Shear Modulus

Shear Modulus (G) is the ratio of shearing stress to the corresponding shearing strain. Another name for shear stress is the Modulus of Rigidity.

$\therefore$ G = $\frac{shearingstress(\sigma_s)}{shearingstrain}$

$\Rightarrow$ G = $\frac{\frac{F}{A}}{\frac{\Delta x}{L}}$

= $\frac{F \times L}{A \times \Delta x}$ … (4)

We also know that, Shearing strain = θ

$\therefore$ G = $\frac{\frac{F}{A}}{\theta}$

= $\frac{F}{A \times \theta}$ … (5)

Further, the shearing stress $\sigma_s$ can also be expressed as

$\sigma_s = G \times \theta$ … (6)

Also, the SI unit of shear modulus is $N/m^2$ or Pa. The shear moduli of a few common materials are given in the table below.

| Material | Shear Modulus (G) $10^9$ N/m$^2$ |
|---|---|
| Aluminum | 25 |
| Brass | 36 |
| Copper | 42 |
| Glass | 23 |
| Iron | 70 |
| Lead | 5.6 |
| Nickel | 77 |
| Steel | 84 |
| Tungsten | 150 |
| Wood | 10 |

From the table, you can observe that the shear modulus is less than Yo for the same materials. Usually, G $\approx \frac{Y}{3}$.

## Elastic Moduli – Bulk Modulus

We have already studied that when we submerge a body in a fluid, it undergoes a hydraulic stress which decreases the volume of the body, leading to a volume strain. Bulk modulus (B) is the ratio of hydraulic stress to the corresponding hydraulic strain.

$$B = -\frac{p}{\left(\frac{\Delta V}{V}\right)} \quad \dots (7)$$

The negative sign means that as the pressure increases, the volume decreases. Hence, for any system in equilibrium, B is always positive. The SI unit of the bulk modulus is $N/m^2$ or Pa. The bulk moduli of a few common materials are given in the table below.

| Material | Bulk Modulus (B)<br>$10^9$ $N/m^2$ |
|---|---|
| Aluminum | 72 |
| Brass | 61 |
| Copper | 140 |
| Glass | 37 |
| Iron | 100 |
| Nickel | 260 |
| Steel | 160 |
| Liquids | |
| Water | 2.2 |
| Ethanol | 0.9 |
| Carbon disulfide | 1.56 |
| Glycerine | 4.76 |

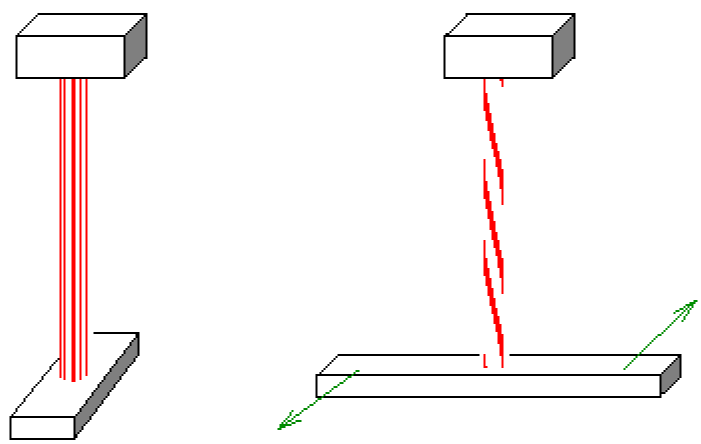
| | |
|---|---|
| Mercury | 25 |
| Gases | |
| Air (at STP) | 1.0 x $10^{-4}$ |

Compressibility (k) is the reciprocal of the bulk modulus. It is the fractional change in volume per unit increase in pressure.

$$\therefore k = \frac{1}{B} = -\frac{1}{\Delta p} \times \frac{\Delta V}{V} \; \ldots (8)$$

From the table, you can observe that the bulk modulus for solids is much larger than that for liquids and gases. Hence, solids are the least compressible while gases are the most compressible. This is because, in solids, there is a tight coupling between the neighboring atoms.

Torsional oscillations



In the Cavendish experiment to measure gravity, we had a quartz fiber dangling from a ceiling. Attached to it was some rod with masses on it. The fiber exerts some torque when the rod is displaced from its equilibrium position.

If small angles, you can say the the torque exerted is proportional to the displacement from equilibrium

$\tau = -\kappa\theta$

This is just like $F = -kx$. $\kappa$ is a constant having to do with the properties of the materials.

So applying $I\alpha = \tau$

$$I\frac{d^2\theta}{dt^2} = -\kappa\theta$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$

Again this is just like , so we have except of $k/m$ we have here $\kappa/I$.

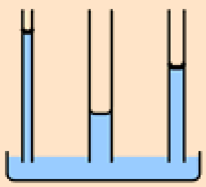
$$\omega^2 = \frac{\kappa}{I}$$

So the quartz fiber will oscillate back and forth at this angular frequency.

# Surface Tension

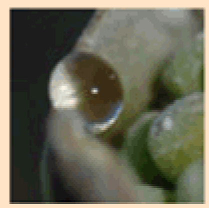
The cohesive forces between liquid molecules are responsible for the phenomenon known as surface tension. The molecules at the surface do not have other like molecules on all sides of them and consequently they cohere more strongly to those directly associated with them on the surface. This forms a surface "film" which makes it more difficult to move an object through the surface than to move it when it is completely submersed.

Surface tension is typically measured in dynes/cm, the force in dynes required to break a film of length 1 cm. Equivalently, it can be stated as surface energy in ergs per square centimeter. Water at 20°C has a surface tension of 72.8 dynes/cm compared to 22.3 for ethyl alcohol and 465 for mercury.

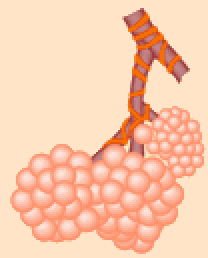


Surface tension and capillarity

Surface tension and bubbles



Surface tension and droplets



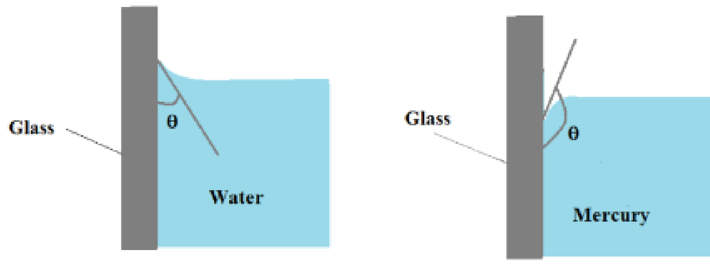
Alveoli of lungs

Other surface tension examples.

# Angle of Contact

## Introduction:

- When water is taken in a glass vessel, the free surface of the water near the walls is curved concave upward. If mercury is taken in a glass vessel, the free surface of mercury near the walls is convex upwards.
- When the liquid is in contact with solid, the angle between the solid surface and the tangent to the free surface of liquid at the point of contact, measured from inside the liquid is called the angle of contact.




- When the liquid surface is curved concave upwards, the angle of contact is acute and when the liquid surface is curved convex upwards, the angle of contact is obtuse.

## Characteristics of the Angle of Contact:

- The angle of contact is constant for a given liquid-solid pair.
- When the angle of contact between the liquid and a solid surface is small (acute), the liquid is said to wet the surface. Thus water wets glass.
- If the angle of contact is large the surface is not wetted. Mercury does not wet glass.
- If there are impurities in liquid, then they alter the values of angle of contact.
- The angle of contact decreases with increase in temperature.
- For a liquid which completely wets the solid, the angle of contact is equal to zero.

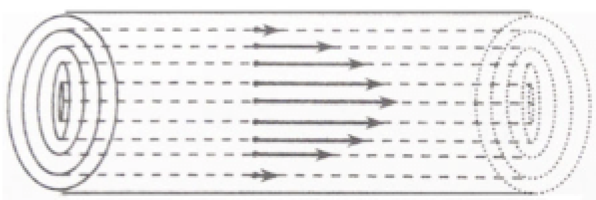
# Viscosity

**Have you ever noticed that some liquids like water flow very rapidly while some others like castor oil do not flow fast? Why is it so? Didn't that question occur to you yet? Well, if it did, we have the answer to it! This is the concept of Viscosity. In this chapter, we will study all about the topic and look at the laws and examples of the same.**

## Viscosity

It is the internal resistance to flow possessed by a liquid. The liquids which flow slowly, have high internal resistance. This is because of the strong intermolecular forces. Therefore, these liquids are more viscous and have high viscosity.

The liquids which flow rapidly have a low internal resistance. This is because of the weak intermolecular forces. Hence, they are less viscous or have low viscosity.



## Laminar Flow

Consider a liquid flowing through a narrow tube. All parts of the liquids do not move through the tube with the same velocity. Imagine the liquid to be made up of a large number of thin cylindrical coaxial layers. The layers which are in contact with the walls of the tube are almost stationary. As we move from the wall towards the centre of the tube, the velocity of the cylindrical layers keeps on increasing till it is maximum at the centre.

This is a laminar flow. It is a type of flow with a regular gradation of velocity in going from one layer to the next. As we move from the centre towards the walls, the velocity of the layers keeps on decreasing. In other words, every layer offers some resistance or friction to the layer immediately below it.

Viscosity is the force of friction which one part of the liquid offers to another part of the liquid. The force of friction f between two layers each having area A sq cm, separated by a distance dx cm, and having a velocity difference of dv cm/sec, is given by:

$f \propto A\ (dv / dx)$

$f = \eta A\ (dv/dx)$

where $\eta$ is a constant known as the coefficient of viscosity and dv/dx is called velocity gradient. If dx =1 , A = 1 sq cm; dv = 1 cm/sec, then $f = \eta$. Hence the coefficient of viscosity may be defined as the force of friction required to maintain a velocity difference of 1 cm/sec between two parallel layers, 1 cm apart and each having an area of 1 sq cm.

### Units of Viscosity

We know that: $\eta = f.dx / A.dv$. Hence, $\eta$ = dynes × cm / $cm^2$ ×cm/sec. Therefore we may write: $\eta$ = dynes $cm^{-2}$ sec or the units of viscosity are dynes sec $cm^{-2}$. This quantity is called 1 Poise.

$f = m \times a$
$\eta = (m \times a \times dx) / (A.dv)$
Hence, $\eta = g\ cm^{-1}\ s^{-1}$
Therefore, $\eta$ = 1 poise

In S.I. units, $\eta = f.dx / A.dv$
$= N \times m / (m^2 \times ms^{-1})$
Therefore we may write, $\eta = N m^{-2}$ or Pas

1 Poise = 1 g $cm^{-1}s^{-1}$ = 0.1 kg $m^{-1}$ $s^{-1}$

## Solved Examples For You

Q: The space between two large horizontal metal plates 6 cm apart, is filled with a liquid of viscosity 0.8N/m. A thin plate of surface area $0.01m^2$ is moved parallel to the length of the plate such that the plate is at a distance of 2m from one of the plates and 4cm from the other. If the plate moves with a constant speed of $1ms^{-1}$, then:

A. Fluid layer with the maximum velocity lies midway between the plates.
B. The layer of the fluid, which is in contact with the moving plate, has the maximum velocity.
C. That layer which is in contact with the moving plate and is on the side of the farther plate is moving with maximum velocity.
D. Fluid in contact with the moving plate and which is on the side of the nearer plate is moving with maximum velocity.

Solution: B) The two horizontal plates are at rest. Also, the plate in between the two plates, is moving ahead with a constant speed of $1ms^{-1}$. The layer closest to this plate will thus move with the maximum velocity.

# Steady flow

**Steady flow is the flow in low speed such that its adjacent layers slide smoothly with respect to each other , Streamline is an imaginary line shows the path of any part of the fluid during its steady flow inside the tube , The density of the streamlines at a point is the number of streamlines crossing perpendicular a unit area point .**

## Characteristics of the streamlines

1. **Imaginary lines do not intersect .**
2. **The tangent at any point along the streamline determines the direction of the instantaneous velocity of each particle of the liquid at that point .**
3. **The number of streamlines does not change as the cross-section area changes , while the streamlines density at a point changes as the cross-section area changes and expresses the flow velocity of the liquid at that point .**
4. **Therefore , streamlines cram up at points of high velocity ( its density increases ) and keep apart at points of low velocity ( its density decreases ) , This means that speed of fluid at any point inside the tube is directly proportional to the density of streamlines at that point .**

### *Conditions of the steady flow*

1. **Liquid should fill the tube completely .**
2. **Speed of the liquid at a certain point in the tube is constant and does not change as the time passes .**
3. **Flow is irrotational , there is no vertex motion .**
4. **No frictional forces between the layers of the nonviscous liquid .**
5. **Flow such that the amount of liquid entering the tube equals that emerging out of it in the same period of time because the liquid is incompressible .**

**Flow rate is the quantity of liquid flowing through a certain cross-sectional area of a tube in one second , Flow rate could be volume flow rate and mass flow rate .**
**Volume flow rate ( $Q_v$ ) is the volume of fluid flowing through a certain area in one second , measuring unit is $m^3/s$ , When volume rate of a liquid = 0.05 $m^3/s$ , It means that volume of fluid flowing through a certain area in one second = 0.05 $m^3$ .**
**Mass flow rate ( $Q_m$ ) is the mass of fluid flowing through a certain area in one second , measuring unit is kg/s , when mass flow rate of a liquid = 3 kg/s , It means that mass of fluid flowing through a certain area in one second = 3 kg .**

### Calculating the flow rate at any cross-sectional area :

**Considering a quantity of liquid of density ( $\rho$ ) , volume ( $V_{ol}$ ) and mass ( m ) flowing in speed ( v ) to move a distance ( $\Delta x$ ) in time ( $\Delta t$ ) through cross-sectional area of the tube ( A ) .**

**From the definition of the volume flow rate :**

**$Q_v = \Delta V_{ol} / \Delta t$**

**$\Delta V_{ol} = A\ \Delta x = A\ v\ \Delta t$ , where $\Delta x = v\ \Delta t$**

**$\therefore Q_v = ( A\ v\ \Delta t ) / \Delta t$**

**$Q_v = A\ v$**

**From the definition of the mas flow rate :**

**$Q_m = \Delta m / \Delta t$**

**$\Delta m = \rho\ \Delta V_{ol}$**

**$\Delta V_{ol} = A\ \Delta x = A\ v\ \Delta t$**

**$Q_m = ( \rho\ A\ v\ \Delta t ) / \Delta t$**

**$Q_m = \rho\ A\ v = \rho\ Q_v$**

**The amount of liquid entering the tube = that emerging out of it in the same period of time .**

**Flow rate ( volume or mass ) is constant at any cross-sectional area and this is called law of conservation of mass that leads to the continuity equation .**

## Deduction of the continuity equation ( relation between flow speed of liquid and cross-sectional area of the tube )

**Imagine that a tube has a fluid in a steady flow where the previous conditions of steady flow are verified .**

**Consider two-cross sectional areas ( $A_1$ , $A_2$ ) perpendicular to the streamlines :**

**At first cross-sectional area ( $A_1$ ) , the speed of liquid through it ( $v_1$) then :**

**The volume flow rate : $Q_v = A_1\ v_1$ , The mass flow rate : $Q_m = \rho\ A_1\ v_1$**

**At second cross-sectional area ( $A_2$) , the speed of liquid through it ( $v_2$ ) then :**

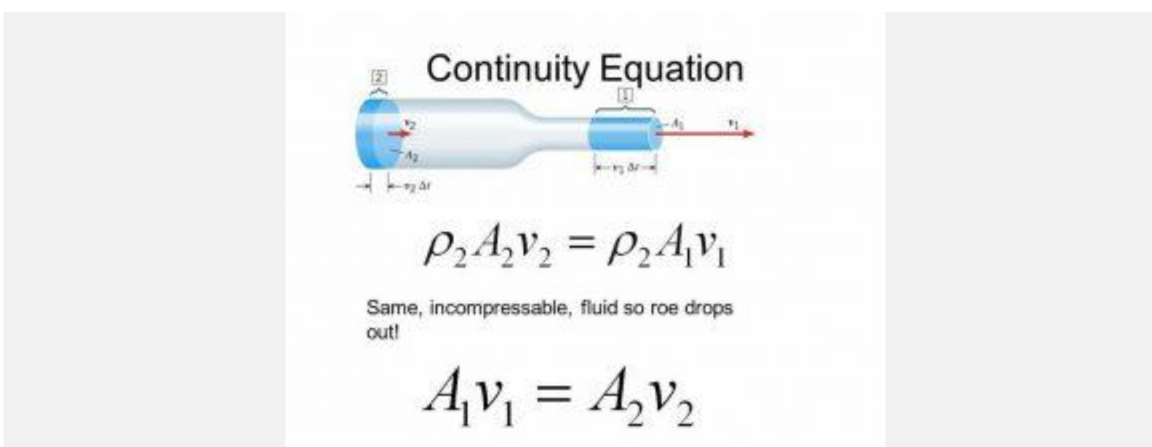
**The volume flow rate : $Q_v = A_2\ v_2$ , the mass flow rate : $Q_m = \rho\ A_2\ v_2$**

**The flow rate ( volume or mass ) is constant in case of steady flow .**

**$\rho\ A_1\ v_1 = \rho\ A_2\ v_2$**

**$A_1\ v_1 = A_2\ v_2$**

**$v_1 / v_2 = A_2 / A_1$ , this relation is called the continuity equation**



## *The continuity equation*

**The velocity of a fluid in a steady flow at any point is inversely proportional to cross-sectional area of the tube at that point .**

**Based on the previous relation ( $A_1 v_1 = A_2 v_2$ ) if :**

**The tube is cylindrical having two cross-sectional area one is wide and the other narrow .**

**$A_1 v_1 = A_2 v_2$**

**$r^2_1 v_1 = r^2_2 v_2$**

**The tube is branched into ( n ) branches of the same cross-sectional area .**

**$A_1 v_1 = n A_2 v_2$**

**$r^2_1 v_1 = n r^2_2 v_2$**

**The tube is branched into number of branches of different cross-sectional area**

**$A_1 v_1 = A_2 v_2 + A_3 v_3 + A_4 v_4$**

**$r^2_1 v_1 = r^2_2 v_2 + r^2_3 v_3 + r^2_4 v_4$**

**Where : $A = \pi r^2$ , r = radius of the tube .**

**The speed is inversely proportional to the cross-sectional area ( $v \propto 1/A$ ) , so , the liquid flows slowly in the tube when its cross-sectional area is big and vice versa .**

### Applications on the continuity equation

**Flow of blood is faster in the main artery than in the blood capillaries because the sum of cross-sectional areas of blood capillaries is greater than the cross-sectional area of the main artery and since ( $v \propto 1/A$ ) , so , speed of blooddecreases in the blood capillaries to allow exchange of oxygen and carbon dioxide gases in the tissues to supply it with food .**

**Design of the gas opening in the stoves , Opening are small so that the gas rushes fast out of it in a high speed ( $v \propto 1/A$ ) .**

### Turbulent flow

**The turbulent flow is the flow when the speed of the fluid exceeds a certain limit and is characterized by small eddy currents , The steady flow of a fluid ( liquid or gas ) becomes turbulent flow if :**

1. **The speed of the fluid exceeded a certain limit and is characterized by small eddy currents .**
2. **A gas transfers from small space to a wider space .**
3. **A gas becomes turbulent when it transfers from high pressure to low pressure .**

**Applications on the pressure at a point ( Connected vessels , U-shaped tube & Mercuric barometer )**

**Applications on pascal's principle , Manometer types and uses**

**Factors affecting the force of viscosity and Applications on the viscosity**

**What is a Reynolds's Number?**

Reynolds's number is a dimensionless quantity that is used to determine the type of flow pattern as laminar or turbulent while flowing through a pipe. Reynolds's number is defined by the ratio of inertial forces to that of viscous forces.

It is given by the following relation:

ReynoldsNumber=InertialForceViscousForce
Re = ρVD/μ

Where,

Re is the Reynolds's number

ρ is the density of the fluid

V is the velocity of flow

D is the pipe diameter

μ is the viscosity of the fluid

If the Reynolds's number calculated is high (greater than 2000), then the flow through the pipe is said to be turbulent. If Reynolds's number is low (less than 2000), the flow is said to be laminar. Numerically, these are acceptable values, although in general the laminar and turbulent flows are classified according to a range. Laminar flow falls below Reynolds's number of 1100 and turbulent falls in a range greater than 2200.

Laminar flow is the type of flow in which the fluid travels smoothly in regular paths. Conversely, turbulent flow isn't smooth and follows an irregular maths with lots of mixing.

An illustration depicting laminar and turbulent flow is given below.

The Reynolds's number is named after the British physicist Osborne Reynolds's. He discovered this while observing different fluid flow characteristics like flow a liquid through a pipe and motion of an airplane wing through the air. He also observed that the type of flow can transition from laminar to turbulent quite suddenly.

Try the following application based problem to understand this concept.

**Problem 1- Calculate** Reynolds's number, if a fluid having viscosity of 0.4 Ns/m2 and relative density of 900 Kg/m3 through a pipe of 20mm with a velocity of 2.5 2.5 m/

**Solution 1 –** Given that,

Viscosity of fluid μ

$\mu = \frac{0.4Ns}{m^2}$

Density of fluid ρ

$\rho = 900Kg/m^2$

Diameter of the fluid

$L = 20 \times 10^{-3}m$

$R_e = \frac{\rho VL}{\mu}$

$= \frac{900 \times 2.5 \times 20 \times 10^{-3}}{0.4}$

$= 112.5$

From the above answer, we observe that the Reynolds number value is less than 2000. Therefore, the flow of liquid is laminar.

For more concepts in Physics, check out our YouTube Channel with loads of video modules to help you out only at BYJU'S.

# What is Bernoulli's equation?

This equation will give you the powers to analyze a fluid flowing up and down through all kinds of different tubes.

## What is Bernoulli's principle?

Bernoulli's principle is a seemingly counterintuitive statement about how the speed of a fluid relates to the pressure of the fluid. Many people feel like Bernoulli's principle shouldn't be correct, but this might be due to a misunderstanding about what Bernoulli's principle actually says. Bernoulli's principle states the following,

**Bernoulli's principle: Within a horizontal flow of fluid, points of higher fluid speed will have less pressure than points of slower fluid speed.**

[Why does it have to be horizontal?]

So within a horizontal water pipe that changes diameter, regions where the water is moving fast will be under less pressure than regions where the water is moving slow. This sounds counterintuitive to many people since people associate high speeds with high pressures. But, we'll show in the next section that this is really just another way of saying that water will speed up if there's more pressure behind it than in front of it. In the section below we'll derive Bernoulli's principle, show more precisely what it says, and hopefully make it seem a little less mysterious.

## How can you derive Bernoulli's principle?

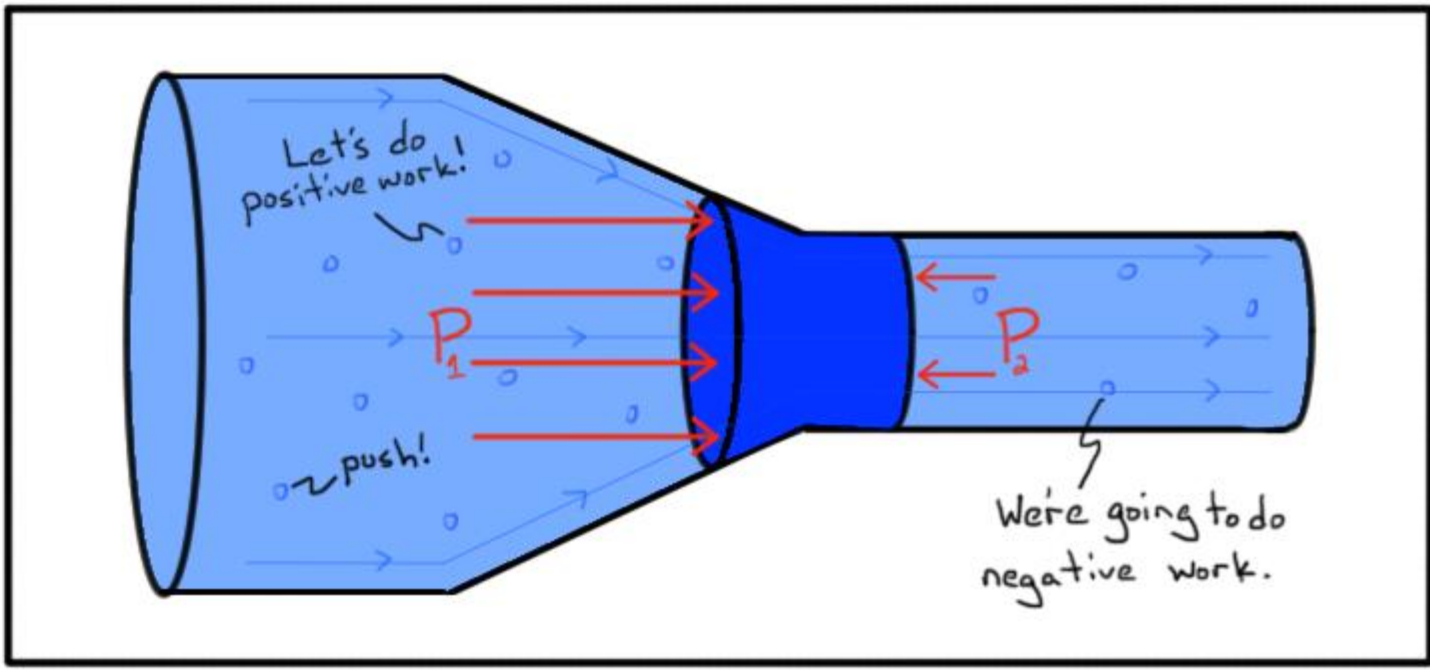
Incompressible fluids have to speed up when they reach a narrow constricted section in order to maintain a constant volume flow rate. This is why a narrow nozzle on a hose causes water to speed up. But something might be bothering you about this phenomenon. If the water is speeding up at a constriction, it's also gaining kinetic energy. Where is this extra kinetic energy coming from? The nozzle? The pipe?

The only way to give something kinetic energy is to do work on it. This is expressed by the work energy principle.

$$W_{external} = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

So if a portion of fluid is speeding up, something external to that portion of fluid must be doing work it. What force is causing work to be done on the fluid? Well, in most real world systems there are lots of dissipative forces that could be doing negative work, but we're going to assume for the sake of simplicity that these viscous forces are negligible and we have a nice continuous and perfectly laminar (streamline) flow. Laminar (streamline) flow means that the fluid flows in parallel layers without crossing paths. In laminar streamline flow there is no swirling or vortices in the fluid. OK, so we'll assume we have no loss in energy due to dissipative forces. In that case, what non-dissipative forces could be doing work on our fluid that cause it to speed up? The pressure from the surrounding fluid will be causing a force that can do work and speed up a portion of fluid.

Consider the diagram below which shows water flowing along streamlines from left to right. As the outlined volume of water enters the constricted region it speeds up. The force from pressure P_1$P_1$ on the left side of the shaded water pushes to the right and does positive work since it pushes in the same direction as the motion of the shaded fluid. The force from pressure P_2$P_2$ on the right side of the shaded fluid pushes to the left and does negative work since it pushes in the opposite direction as the motion of the shaded fluid.




We know that the water must speed up (due to the continuity equation) and therefore have a net positive amount of work done on it. So the work done by the force from pressure on the left side must be larger than the amount of negative work done by the force from pressure on the right side. This means that the pressure on the wider/slower side P_1$P_1$ has to be larger than the pressure on the narrow/faster side P_2$P_2$.

> This inverse relationship between the pressure and speed at a point in a fluid is called **Bernoulli's principle**.
>
> **Bernoulli's principle:** At points along a horizontal streamline, higher pressure regions have lower fluid speed and lower pressure regions have higher fluid speed.

It might be conceptually simplest to think of Bernoulli's principle as the fact that a fluid flowing from a high pressure region to a low pressure region will accelerate due to the net force along the direction of motion.

The idea that regions where the fluid is moving fast will have lower pressure can seem strange. Surely, a fast moving fluid that strikes you must apply more pressure to your body than a slow moving fluid, right? Yes, that is right. But we're talking about two different pressures now. The pressure that Bernoulli's principle is referring to is the internal fluid pressure that would be exerted in all directions during the flow, including on the sides of the pipe. This is different from the pressure a fluid will exert on you if you get in the way of it and stop its motion.

[I still don't get the difference.]

Note that Bernoulli's principle does not say that a fast moving fluid *can't* have significantly high pressures. It just says that the pressure in a slower region of that same flowing system must have even larger pressure than the faster moving region.

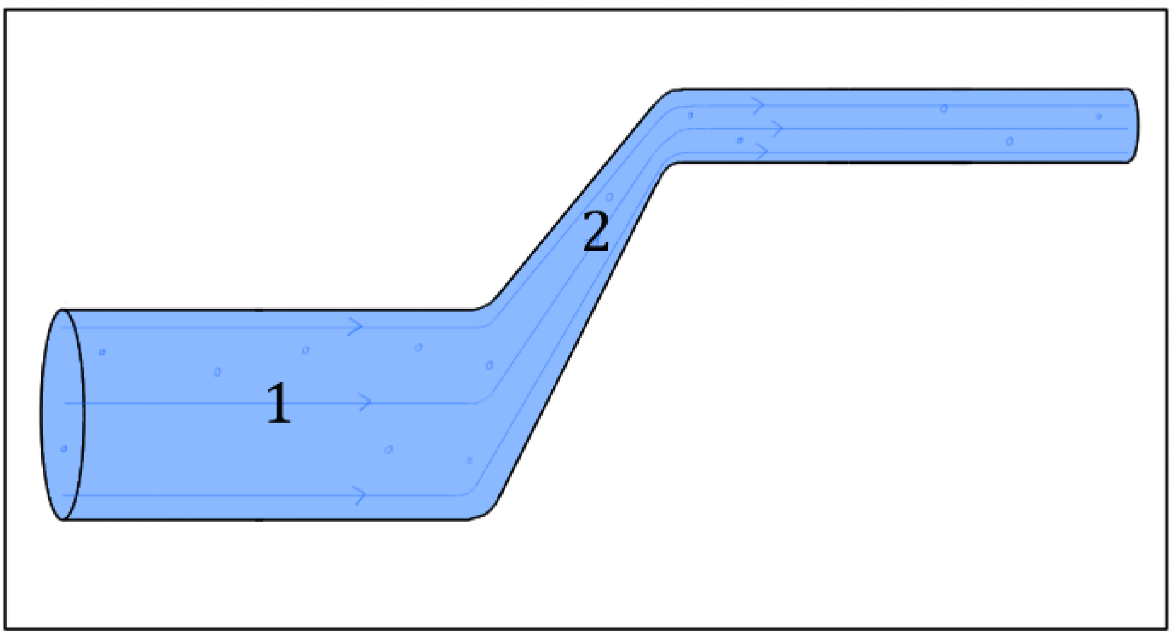
# What is Bernoulli's equation?

Bernoulli's equation is essentially a more general and mathematical form of Bernoulli's principle that also takes into account changes in gravitational potential energy. We'll derive this equation in the next section, but before we do, let's take a look at Bernoulli's equation and get a feel for what it says and how one would go about using it.

Bernoulli's equation relates the pressure, speed, and height of any two points (1 and 2) in a steady streamline flowing fluid of density $\rho$. Bernoulli's equation is usually written as follows,

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

The variables $P_1$, $v_1$, $h_1$ refer to the pressure, speed, and height of the fluid at point 1, whereas the variables $P_2$, $v_2$, and $h_2$ refer to the pressure, speed, and height of the fluid at point 2 as seen in the diagram below. The diagram below shows one particular choice of two points (1 and 2) in the fluid, but Bernoulli's equation will hold for any two points in the fluid.



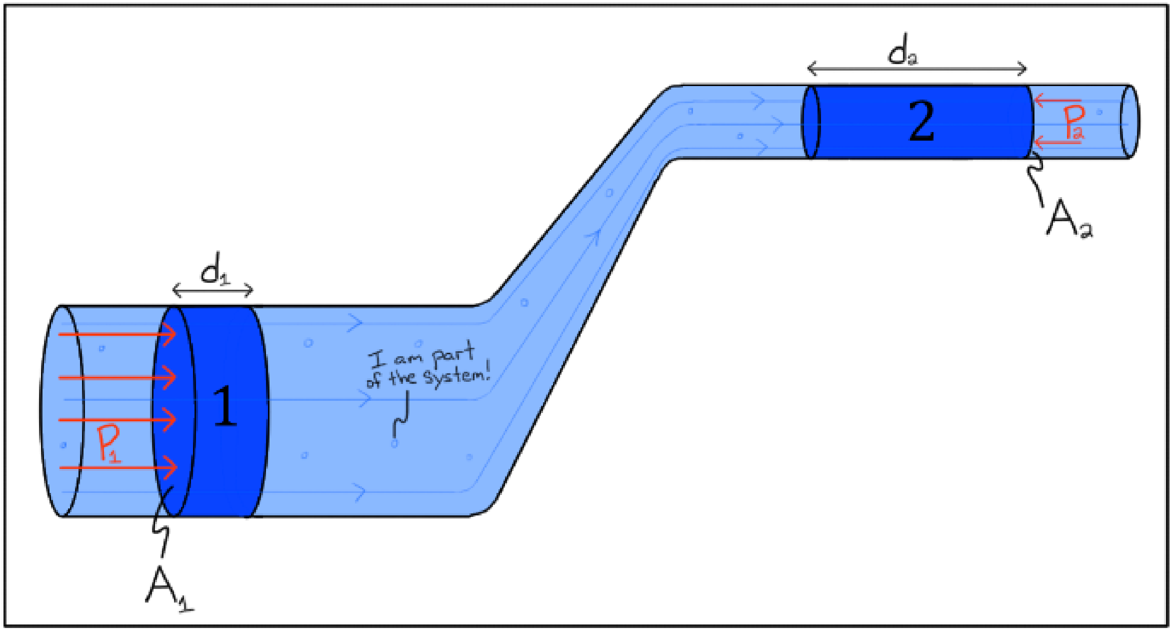
When using Bernoulli's equation, how do you know where to choose your points? Choosing one of the points at the location where you want to find an unknown variable is a must. Otherwise how will you ever solve for that variable? You will typically choose the second point at a location where you have been given some information, or where the fluid is open to the atmosphere, since the absolute pressure there is known to be atmospheric pressure $P_{atm} = 1.01 \times 10^5 Pa$.

Note that the $h$ refers to the height of the fluid above an arbitrary level that you can choose in any way that is convenient. Typically it is often easiest to just choose the lower of the two points (1 or 2) as the height where $h = 0$. The $P$ refers to the pressure at that point. You can choose to use gauge pressure or absolute pressure, but whichever kind of pressure you choose (gauge or absolute) must also be used on the other side of the equation. You can't insert the gauge pressure at point 1, and the absolute pressure at point 2. Similarly, if you insert the gauge pressure at point 1 and solve for the pressure at point 2, the value you obtain will be the gauge pressure at point 2 (not the absolute pressure).

The terms $\frac{1}{2}\rho v^2$ and $\rho gh$ in Bernoulli's equation look just like kinetic energy $\frac{1}{2}mv^2$ and potential energy $mgh$, only with the mass $m$ replaced with density $\rho$. So it may not come as much of a surprise that Bernoulli's equation is the result of applying conservation of energy to a flowing fluid. We'll derive Bernoulli's equation using conservation of energy in the next section.

## How do you derive Bernoulli's equation?

Consider the following diagram where water flows from left to right in a pipe that changes both area and height. As before, water will speed up and gain kinetic energy $K$ at constrictions in the pipe, since the volume flow rate must be maintained for an incompressible fluid even if those constricted sections move upward. But now since the constriction also causes the fluid to move upward, the water will be gaining gravitational potential energy $U_g$ as well as kinetic energy $K$. We will derive Bernoulli's equation by setting the energy gained by the fluid equal to the external work done on the fluid.




Let's assume the energy system we're considering is composed of the volumes of water 1 and 2 as well as all the fluid in between those volumes. If we assume the fluid flow is streamline, non-viscous, and there are no dissipative forces affecting the flow of the fluid, then any extra energy $\Delta(K+U)_{system}$ added to the system will be caused by the external work $(W_{external})$ done on the fluid from pressure forces surrounding it. [Doesn't gravity do work too?]

We can express this mathematically as,

$$W_{external} = \Delta(K+U)_{system}$$

First we'll try to find the external work done $W_{external}$ on the water. None of the water between points 1 and 2 can do external work since that water is all part of our energy system. The only pressures that can directly do external work on our system are $P_1$ and $P_2$ as shown in the diagram. The water at $P_1$ to the left of volume 1 will do positive work since the force points in the same direction as the motion of the fluid. The water at $P_2$ to the right of volume 2 will do negative work on our system since it pushes in the opposite direction as the motion of the fluid.

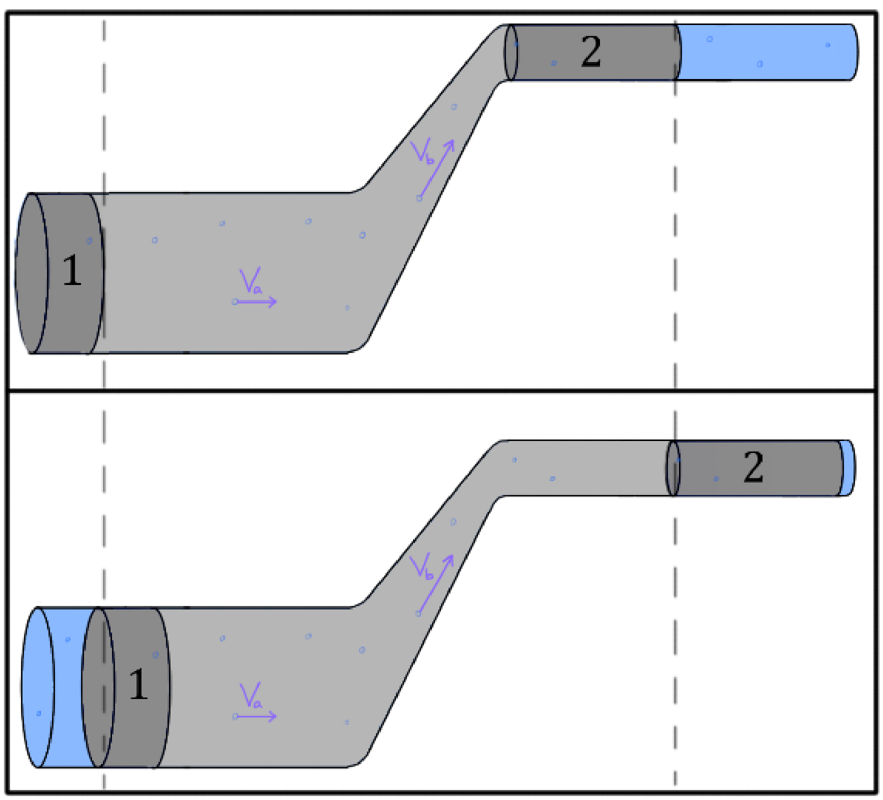
For simplicity's sake we'll consider the case where the force from water pressure to the left of volume 1 pushes volume 1 through its entire width $d_1$. Assuming the fluid is incompressible, this must displace an equal volume of water everywhere in the system, causing volume 2 to be displaced through its length a distance $d_2$.

> Work can be found with $W = Fd$. We can plug in the formula for the force from pressure $F = PA$ into the formula from work to get $W = PAd$. So, the positive work done on our system by the water near point 1 will be $W_1 = P_1 A_1 d_1$ and the work done by the water near point 2 will be $W_2 = -P_2 A_2 d_2$. [How do you determine the signs here?]
>
> Plugging these expressions for work into the left side of our work energy formula $W_{net} = \Delta(K + U)_{system}$ we get,
>
> $$P_1 A_1 d_1 - P_2 A_2 d_2 = \Delta(K + U)_{system}$$
>
> But the terms $A_1 d_1$ and $A_2 d_2$ have to be equal since they represent the volumes of the fluid displaced near point 1 and point 2. If we assume the fluid is incompressible, an equal volume of fluid must be displaced everywhere in the fluid, including near the top. So, $V_1 = A_1 d_1 = A_2 d_2 = V_2$. We can just write the volume term simply as $V$ since the volumes are equal. This simplifies the left side of the work energy formula to,
>
> $$P_1 V - P_2 V = \Delta(K + U)_{system}$$

That takes care of the left hand side. Now we have to address the right hand side of this equation. This is a crucial and subtle part of the derivation. Remember that our system includes not only the shaded portions of water near point 1 and 2, but also all the water in between those two points. How will we ever be able to account for all the change in kinetic energy and gravitational potential energy of all parts of that large and winding system?

Well, we have to make one more assumption to finish the derivation. We're going to assume that the flow of the fluid is steady. By "steady flow" we mean that the speed of the fluid passing by a particular point in the pipe doesn't change. In other words, if you stood and stared at any one particular section of the transparent pipe, you would see new water moving past you every moment, but if there's steady flow, then all the water would have the same speed when it moves past that particular point.

So how does the idea of steady flow help us figure out the change in energy of the big winding system of fluid? Consider the diagram below. Our energy system consists of the greyed out fluid (volume 1, volume 2, and all fluid in between). In the first image, the system has some amount of total energy $(K+U)_{initial}$. In the second image the entire system had work done on it, gains energy, shifts to the right, and now has some different total energy $(K+U)_{final}$. But notice that the energy of the fluid between the dashed lines will be the same as it was before the work was done assuming a steady flow. Water changed position and speed in the region between the dashed lines, but it did so in such a way that it will be moving with the exact same speed (e.g. $v_a$ and $v_b$) and have the same height as the previous water had in that location. The only thing that's different about our system is that volume 2 now extends into a section of the pipe it wasn't in previously, and now nothing in our system is occupying the old position behind volume 1.



Overall this means that the total change in the energy of the system can be found by simply considering the energies of the end points. Namely, we can take the kinetic and potential energy $(K_2 + U_2)$ that now exists in volume 2 after the work was done and subtract the kinetic and potential energy $(K_1 + U_1)$ that no longer exists behind volume 1 after the work was done. In other words, $\Delta(K+U)_{system} = (K_2 + U_2) - (K_1 + U_1)$. [I still don't get it.]

Plugging this into the right hand side of the work energy formula $P_1V - P_2V = \Delta(K+U)_{system}$ we get,

$$P_1V - P_2V = (K_2 + U_2) - (K_1 + U_1)$$

Now we'll substitute in the formulas for kinetic energy $K = \frac{1}{2}mv^2$ and gravitational potential energy $U_g = mgh$ to get,

$$P_1V - P_2V = (\frac{1}{2}m_2v_2^2 + m_2gh_2) - (\frac{1}{2}m_1v_1^2 + m_1gh_1)$$

In this equation $P_1$ and $P_2$ represent the pressures of the fluid in volumes 1 and 2 respectively. The variables $v_1$ and $v_2$ represent the speeds of the fluid in volumes 1 and 2 respectively. And $h_1$ and $h_2$ represent the height of the fluid in volumes 1 and 2 respectively.

But since we are assuming the fluid is incompressible, the displaced masses of volumes 1 and 2 must be the same $m_1 = m_2 = m$. So getting rid of the subscript on the $m$'s we get,

$$P_1V - P_2V = (\frac{1}{2}mv_2^2 + mgh_2) - (\frac{1}{2}mv_1^2 + mgh_1)$$

We can divide both sides by $V$ and drop the parenthesis to get,

$$P_1 - P_2 = \frac{\frac{1}{2}mv_2^2}{V} + \frac{mgh_2}{V} - \frac{\frac{1}{2}mv_1^2}{V} - \frac{mgh_1}{V}$$

We can simplify this equation by noting that the mass of the displaced fluid divided by volume of the displaced fluid is the density of the fluid $\rho = \frac{m}{V}$. Replacing $\frac{m}{V}$ with $\rho$ we get,

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2 + \rho g h_2 - \frac{1}{2}\rho v_1^2 - \rho g h_1$$

Now, we're just going to rearrange the formula using algebra to put all the terms that refer to the same point in space on the same side of the equation to get,

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

And there it is, finally. This is Bernoulli's equation! It says that if you add up the pressure $P$ plus the kinetic energy density $\frac{1}{2}\rho v^2$ plus the gravitational potential energy density $\rho g h$ at any 2 points in a streamline, they will be equal.

Bernoulli's equation can be viewed as a conservation of energy law for a flowing fluid. We saw that Bernoulli's equation was the result of using the fact that any extra kinetic or potential energy gained by a system of fluid is caused by external work done on the system by another non-viscous fluid. You should keep in mind that we had to make many assumptions along the way for this derivation to work. We had to assume streamline flow and no dissipative forces, since otherwise there would have been thermal energy generated. We had to assume steady flow, since otherwise our trick of canceling the energies of the middle section would not have worked. We had to assume incompressibility, since otherwise the volumes and masses would not necessarily be equal.

Since the quantity $P + \frac{1}{2}\rho v^2 + \rho g h$ is the same at every point in a streamline, another way to write Bernoulli's equation is,

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$

This constant will be different for different fluid systems, but for a given steady state streamline non-dissipative flowing fluid, the value of $P + \frac{1}{2}\rho v^2 + \rho g h$ will be the same at any point along the flowing fluid.

## How is Bernoulli's principle a result of Bernoulli's equation?

We should note here that Bernoulli's principle is contained within Bernoulli's equation. If we start with,

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

and assume that there is no change in the height of the fluid, the $\rho g h$ terms cancel if we subtract them from both sides. [How?]

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Or we could write it as,

$$P + \frac{1}{2}\rho v^2 = \text{constant}$$

This formula highlights Bernoulli's principle since if the speed $v$ of a fluid is larger in a given region of streamline flow, the pressure $P$ must be smaller in that region (which is Bernoulli's principle). An increase in speed $v$ must be accompanied by a simultaneous decrease in the pressure $P$ in order for the sum to always add up to the same constant number.

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# UNIT-4

# Simple Harmonic Motion

We see different kinds of motion every day. The motion of the hands of a clock, motion of the wheels of a car, etc. Did you ever notice that these types of motion keep repeating themselves? Such motions are periodic in nature. One such type of periodic motion is simple harmonic motion (S.H.M.). But what is S.H.M.? Let's find out.

Introduction to Periodic Motion H

SHM as a projection of circular motion I H

VST Simple Harmonic Motion Problem 4 and its Solution

## Periodic Motion and Oscillations

A motion that repeats itself in equal intervals of time is *periodic***.** We need to know what periodic motion is to understand simple harmonic motion.

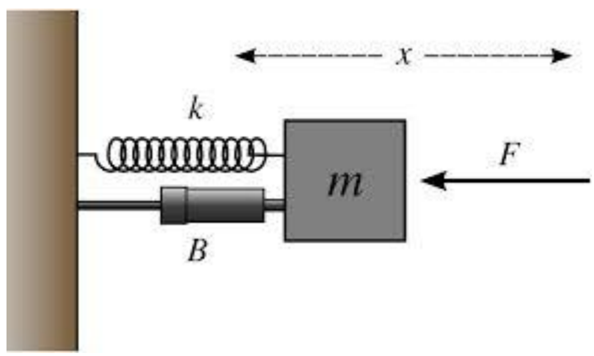
Periodic motion is the motion in which an object repeats its path in equal intervals of time. We see many examples of periodic motion in our day-to-day life. The motion of the hands of a clock is periodic motion. The rocking of a cradle, swinging on a swing, leaves of a tree moving to and fro due to wind breeze, these all are examples of periodic motion.

The particle performs the same set of movement again and again in a periodic motion. One such set of movement is called an Oscillation. A great example of oscillatory motion is Simple Harmonic Motion. Let's learn about it below.

# Simple Harmonic Motion (S.H.M.)

When an object moves to and fro along a line, the motion is called simple harmonic motion**.** Have you seen a pendulum? When we swing it, it moves to and fro along the same line. These are called oscillations. Oscillations of a pendulum are an example of simple harmonic motion.

Now, consider there is a spring that is fixed at one end. When there is no force applied to it, it is at its equilibrium position. Now,

- If we pull it outwards, there is a force exerted by the string that is directed towards the equilibrium position.
- If we push the spring inwards, there is a force exerted by the string towards the equilibrium position.
- 


- In each case, we can see that the force exerted by the spring is towards the equilibrium position. This force is called the restoring force. Let the force be F and the displacement of the string from the equilibrium position be x.

Therefore, the restoring force is given by, $F = -kx$ (the negative sign indicates that the force is in opposite direction). Here, k is the constant called the force constant. Its unit is N/m in S.I. system and dynes/cm in C.G.S. system.

# Linear Simple Harmonic Motion

Linear simple harmonic motion is defined as the linear periodic motion of a body in which the restoring force is always directed towards the equilibrium position or mean position and its magnitude is directly proportional to the displacement from the equilibrium position. All simple harmonic motions are periodic in nature but all periodic motions are not simple harmonic motions.

Now, take the previous example of the string. Let its mass be m. The acceleration of the body is given by,

$$a = F/m = -kx/m = -\omega^2 x$$

### Here, $k/m = \omega^2$ ($\omega$ is the angular frequency of the body)

#### Concepts of Simple Harmonic Motion (S.H.M)

- **Amplitude**: The maximum displacement of a particle from its equilibrium position or mean position is its amplitude. Its S.I. unit is the metre. The dimensions are $[L^1 M^0 T^0]$. Its direction is always away from the mean or equilibrium position.
- **Period**: The time taken by a particle to complete one oscillation is its period. Therefore, period if S.H.M. is the least time after which the motion will repeat itself. Thus, the motion will repeat itself after nT. where n is an integer.
- **Frequency**: Frequency of S.H.M. is the number of oscillations that a particle performs per unit time. S.I. unit of frequency is hertz or r.p.s(rotations per second). Its dimensions are $[L^0 M^0 T^{-1}]$.

- **Phase**: Phase of S.H.M. is its state of oscillation. Magnitude and direction of displacement of particle represent the phase. The phase at the beginning of the motion is known as Epoch(α)

## Difference between Periodic and Simple Harmonic Motion

| Periodic Motion | Simple Harmonic Motion |
|---|---|
| In the periodic motion, the displacement of the object may or may not be in the direction of the restoring force. | In the simple harmonic motion, the displacement of the object is always in the opposite direction of the restoring force. |
| The periodic motion may or may not be oscillatory. | Simple harmonic motion is always oscillatory. |
| Examples are the motion of the hands of a clock, the motion of the wheels of a car, etc. | Examples are the motion of a pendulum, motion of a spring, etc. |

## Solved Questions for You

Q: Assertion(A): In simple harmonic motion, the motion is to and fro and periodic
Reason(R): Velocity of the particle $V = \omega\sqrt{A^2 - x^2}$ where x is displacement as measured from the extreme position
Chose the right answer:

a. Both a and B are true and R is the correct explanation of A.

b. Both A and B are true and R is not the correct explanation of A.

c. A is true and R is false.

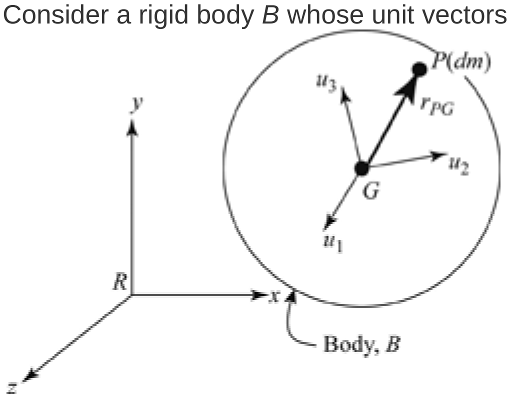
d. A is false and R is true.

Solution: c) A is true and R is false. $V = \omega\sqrt{A^2 - x^2}$ is measured from the mean position. SHM involves to and fro periodic motion.

# Product of Inertia

# Product Of Inertia Of A Mass

For an object rotating about an axis, the resistance of a body to accelerate is called inertia of mass. It is the product of rotating object's mass and square of the span between axis of rotation and mass centre. It has dimensional unit of $ML^2T^0$.
Inertia of mass, $I = mR^2$
Consider a rigid body *B* whose unit vectors and mass centre are depicted in the figure below.




Moment of inertia of a mass about x-axis, $I_{xx} = \int (y^2 + z^2)\, dm$
Moment of inertia of a mass about y-axis, $I_{yy} = \int (x^2 + z^2)\, dm$
Moment of inertia of a mass about z-axis, $I_{zz} = \int (x^2 + y^2)\, dm$
Here *x, y* and *z* are the position vector's components $(\vec{r}_{PG})$
For any rigid body the **product of inertia** is given by
For x-y plane: $I_{xy} = \int (xy)\, dm$
For x-z plane: $I_{xz} = \int (xz)\, dm$
For y-z plane: $I_{yz} = \int (yz)\, dm$
Product of inertia of mass is the symmetric measure for a body. If any one of the three planes is a symmetric plane, then the product of inertia of the perpendicular planes are zero. If X-Y plane is symmetric then $I_{xz} = 0$ and $I_{yz} = 0$
In case of revolution bodies, the body will be symmetric about two axes, hence two planes will be symmetric. In such case the product of inertia, for all three planes is zero. $I_{xy} = I_{xz} = I_{yz} = 0$

# Principal Axes of Inertia

- We've spent the last few lectures deriving the general expressions for $\mathbf{L}$ and $T_{\text{rot}}$ in terms of the inertia tensor
- Both expressions would be a great deal simpler if the inertia tensor was *diagonal*. That is, if:

$$I_{ij} = I_i \delta_{ij}$$

or

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

- Then we could write

$$L_i = \sum_j I_{ij}\omega_j = \sum_j I_i \delta_{ij}\omega_j = I_i\omega_i$$

$$T_{\text{rot}} = \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j = \frac{1}{2}\sum_{i,j} I_i\delta_{ij}\omega_i\omega_j = \frac{1}{2}\sum_i I_i\omega_i^2$$

- We've already seen that the elements of the inertia tensor transform under rotations
- So perhaps we can rotate to a set of axes for which the tensor (for a given rigid body) is diagonal
  - These are called the *principal axes* of the body
  - All the rotational problems you did in first-year physics dealt with rotation about a principal axis – that's why the equations looked simpler.
- If a body is rotating solely about a principal axis (call it the *i* axis) then:

$$L_i = I_i\omega_i \text{, or } \mathbf{L} = I_i\boldsymbol{\omega}$$

- If we can find a set of principal axes for a body, we call the three non-zero inertia tensor elements the *principal moments* of inertia

## Finding the Principal Moments

- In general, it's easiest to first determine the principal moments, and then find the principal axes
- We know that if we're rotating about a principal axis, we have:

$$\mathbf{L} = I\boldsymbol{\omega}$$

A principal moment

- But the general relation $L_i = \sum_j I_{ij}\omega_j$ also holds. So,


- But the general relation $L_i = \sum_j I_{ij}\omega_j$ also holds. So,


$$\begin{aligned} L_1 &= I\omega_1 = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ L_2 &= I\omega_2 = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ L_3 &= I\omega_3 = I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{aligned}$$

- Rearranging the equations gives:

$$(I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0$$
$$I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 = 0$$
$$I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 = 0$$

- Linear algebra fact:
  - We can consider this as a system of equations for the $\omega_i$
  - Such a system has a solution only if the determinant of the coefficients is zero
- In other words, we need:

$$\begin{vmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0$$

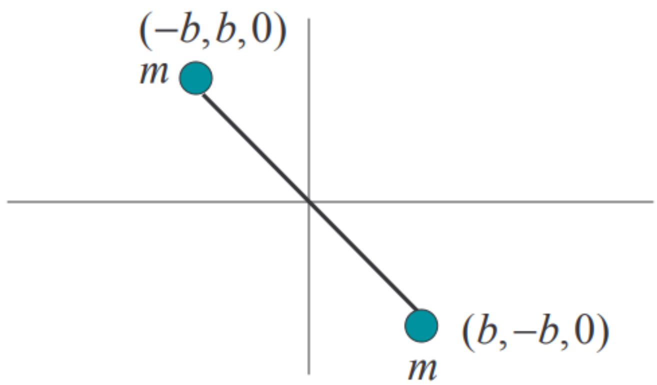
- The determinant results in a cubic equation for $I$
- The three solutions are the three principal moments of inertia for the body (one corresponding to each principal axis)
- And this brings us the resolution of the apparent contradiction between freshman-level physics, in which there were three moments of inertia, and this course, where we needed 6 numbers
  - In the earlier course, only rotations about principal axes were considered!

# Finding the Principal Axes

- Now all that's left to do is find the principal axes. We do this by solving the system of equations for $\omega_i$
  - Using one of the possible values of $I$ – call it $I_1$
  - This will give the direction of the first principal axis
- It turns out that we won't be able to find all three components
  - But we can determine the ratio $\omega_1 : \omega_2 : \omega_3$
  - And that's enough to figure out the direction of the first principle axis (in whatever coordinate system we're using)

# Example: Dumbbell

- Consider the same dumbbell that appeared last lecture, and define the coordinate system as follows:



$$\mathbf{I} = m\begin{bmatrix} 2b^2 & 2b^2 & 0 \\ 2b^2 & 2b^2 & 0 \\ 0 & 0 & 4b^2 \end{bmatrix} = 2b^2 m \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- So the equation we need to solve is:

$$\begin{vmatrix} 1-I & 1 & 0 \\ 1 & 1-I & 0 \\ 0 & 0 & 2-I \end{vmatrix} = 0$$

$$(2-I)\left[(1-I)^2 - 1\right] = 0$$

$$(2-I)[I^2 - 2I] = 0$$

$$I(2-I)(I-2) = 0$$

$$I = (0, 2 \text{ or } 2) \times 2mb^2$$

- Let's find the principal axis associated with $I = 0$:

$$\omega_1 + \omega_2 = 0$$

$$\omega_1 + \omega_2 = 0$$

$$4\omega_3 = 0$$

- So the ratio of the angular momentum components in our coordinate system when the object it rotating about the principal axis with $I = 0$ is:

$$\omega_1 : \omega_2 : \omega_3 = 1 : -1 : 0$$

meaning the axis is defined by the vector:

$$\mathbf{r} = \mathbf{e_x} - \mathbf{e_y}$$

In other words, along the axis of the dumbbell

If an object has an axis of symmetry, that axis is always a principal axis

- What about the other principal axes?
  - The axes associated with $I = 4mb^2$ are:

$$-\omega_1 + \omega_2 = 0$$
$$\omega_1 - \omega_2 = 0$$
$$0 = 0$$

- There's not much information in those equations!
  - For example, the $z$ component could be anything
- This means that *any* two axes perpendicular to the axis of the dumbbell can be taken as principal axes
- Note that the principal axes one finds can depend both on the shape of the body *and* on the point about which it's rotating

## When Can We Find Principal Axes?

- We can always write down the cubic equation that one must solve to determine the principal moments
- But if we want to interpret these as physically meaningful quantities, the roots of that cubic have to be *real*
  - Recall that in general, cubics can have two complex roots
- Fortunately, we're *not* in the general case here
- The inertia tensor is both real and symmetric – in particular, it satisfies:

$$I_{ij} = I_{ij}^*$$

- Matrices that satisfy this restriction are called *Hermitian*
- For such matrices, the principal moments can always be found, and they are always real (see proof in text)

This mathematics will come up again in Quantum Mechanics
Principal Moments $\leftrightarrow$ Eigenvalues
Principal Axes $\leftrightarrow$ Eigenfunctions

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# UNIT-5

## : Relativistic Mechanics

### Introduction

We know that Newton's laws of motion hold and are invariant in inertial frames of reference that are related by the Galilean transformation. But it is the Lorentz transformation rather than the Galilean transformation that is correct. That means that the laws of classical mechanics cannot be correct, and we must find new laws of relativistic mechanics.

In doing so, we will be guided by 3 principles:

1. A correct relativistic law must hold in all inertial frames, i.e., it must be invariant under the Lorentz transformation.
2. Relativistic definitions and laws must reduce to their nonrelativistic counterparts when applied to systems moving much slower than the speed of light.
3. Our relativistic laws must agree with experiment.

### Mass

How do we define mass $m$? We know that at slow speeds, mass can be defined classically by $m = F/a$. In relativity we will measure the mass $m$ of an object in its rest frame. This is called the **rest mass** or the **proper mass**. Observers in different rest frames could measure the mass of an object by bringing the object to rest in their frame of reference, and then measuring its mass. The mass they would measure would be the same in all inertial reference frames.

## Relativisitic Momentum

Classically, momentum is defined by

$$\vec{p} = m\vec{u} \tag{1}$$

where $m$ and $\vec{u}$ the mass and velocity of an object. What is the correct relativistic definition of momentum? We have the freedom to define momentum any way we like, but to be useful, we would like to define it such that total momentum is conserved in all inertial frames.

Recall the law of momentum conservation. If there are $n$ bodies with momenta $\vec{p}_1$, ..., $\vec{p}_n$, then, in the absence of external forces, the total momentum is given by

$$\sum \vec{p} = \vec{p}_1 + ... + \vec{p}_n \tag{2}$$

cannot change. We would like to define momentum relativistically to preserve conservation of momentum.

As your book describes in section 2.3, we can't use the classical definition Eq. (1) because of the way velocity transforms between reference frames. For example, the way $u_y$ transforms in going from $S$ to $S'$ depends on $u_x$. The classical definition of momentum is

$$\vec{p} = m\vec{u} = m\frac{d\vec{r}}{dt} \tag{3}$$

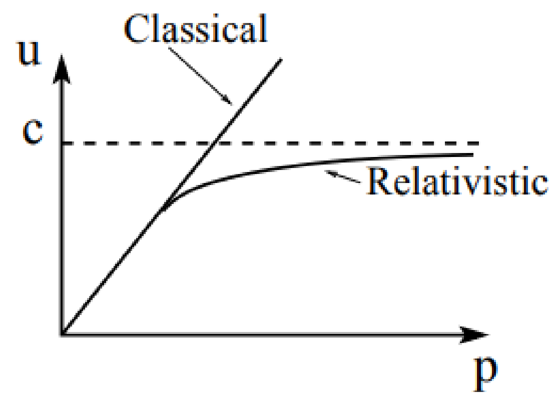



Figure 1: Velocity versus momentum.

The problem is that both $\vec{r}$ and $t$ are subject to the Lorentz transformation and that makes things messy.

The relativistic way to define momentum is

$$\vec{p} = m\frac{d\vec{r}}{dt_0} \tag{4}$$

where $t_0$ is the proper time, i.e., the time measured in the rest frame of the particle or object. Notice that with this definition,

$$p_y = m\frac{dy}{dt_0} \tag{5}$$

which is invariant when one goes between reference frames that are moving relative to one another along the $x-$axis. That is because $t_0$ is invariant and $y = y'$. Similarly for $p_z$.

Since $dt = \gamma dt_0$ where $\gamma = 1/\sqrt{1 - u^2/c^2}$, we can write

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}} = \gamma m\vec{u} \tag{6}$$

A constant force changes the momentum and hence the velocity. Notice that as $u \to c$, $\gamma$ and hence $p$ increase without limit, but the speed $u$ never reaches the speed of light (see plot of $u$ versus $p$ in Fig 2.2 of your book). Thus no object can go faster than the speed of light.

We can define a relativistic mass $M(u)$ by

$$M(u) = \frac{M}{\sqrt{1 - u^2/c^2}} = M\gamma \tag{7}$$

Your book calls this a variable mass $m_{var}$, but it doesn't want to use this. However, it does make it easy to write the momentum in the traditional form

$$\vec{p} = M(u)\vec{u} \tag{8}$$

Notice from Eq. (7) that if the speed $u$ of an object with nonzero mass is equal to the speed of light, the relativistic mass is infinity. If the object's speed exceeds the speed of light, then the relativistic mass $M(u)$ is imaginary. Clearly, this is absurd and so, once again, the velocity of an object cannot exceed the speed of light. If the object has mass, it cannot travel at the speed of light; its speed must be less than the speed of light.

**Relativistic Energy**

Now we need to define a relativistic energy. We will use 2 criteria:

1. When applied to slowly moving bodies, the new definition reduces to the classical definition.

2. The total energy $\sum E$ of an isolated system should be conserved in all reference frames.

The relativistic energy that satisfies these requirements turns out to be

$$E = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = \gamma mc^2 \tag{9}$$

This applies to any single body, no matter how big or how small. Notice that the units are right. (It's always important to check units.) $\gamma$ is dimensionless, and $mc^2$ has units of energy. (Recall that kinetic energy is $(1/2)mv^2$.)

Now let's take the nonrelativistic (slow) limit $u \ll c$. Then

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} \tag{10}$$

Therefore, when $u \ll c$,

$$E = \gamma mc^2 \approx \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right) mc^2 = mc^2 + \frac{1}{2}mu^2 \tag{11}$$

The first term is a constant independent of the speed $u$, and we can always add a constant to the energy since we can set the zero of the energy ($E = 0$) anywhere. The second term is just the usual classical expression for the kinetic energy. So our definition of the relativistic energy satisfies the criterion of reducing to the classical kinetic energy in the nonrelativistic (slow) limit.

Now let's look at that constant $mc^2$. If the object is at rest, then $u = 0$, $\gamma = 1$, and Eq. (9) reduces to the famous equation

$$E = mc^2 \tag{12}$$

This is called the rest energy of the mass $m$. If we could convert a mass $m$ completely into energy, then this is the amount of energy we would get. It's a lot of energy because the speed of light $c$ is so large, and $c^2$ is even larger. For example, the rest energy of a 1 kg lump of metal is

$$E = mc^2 = (1 \text{ kg}) \times (3 \times 10^8 \text{ m/s})^2 = 9 \times 10^{16} \text{ joules} \tag{13}$$

which is about the energy generated by a large power plant in one year. $E = mc^2$ is the basic equation behind nuclear power, and atomic bombs. These are powered by nuclear fission in which, typically, uranium 235 nuclei are split apart, converting about 1/1000 of the rest energy into heat. Thus 1 kg of $^{235}$U can yield a fantastic amount of heat ($9 \times 10^{13}$ J).

When an object is not at rest, its energy $E = \gamma mc^2$ is the sum of its rest energy $mc^2$ plus its kinetic energy $K = (E - mc^2)$:

$$E = mc^2 + K \tag{14}$$

where

$$K = E - mc^2 = (\gamma - 1)mc^2 \tag{15}$$

In the nonrelativistic limit $K \approx \frac{1}{2}mu^2$, the classical kinetic energy. However, in the relativistic limit of $u \to c$, we get something quite different. Because $\gamma$ can approach infinity, $K$ can approach infinity even though the speed $u$ can never reach the speed of light $c$. Notice that $K$, like its classical counterpart, is always positive ($K \geq 0$).

Two Useful Relations

Let's derive 2 useful relations starting from

$$\vec{p} = \gamma m\vec{u} \tag{16}$$

and

$$E = \gamma mc^2 \tag{17}$$

If we divide Eq. (16) by Eq. (17), we obtain

$$\frac{\vec{p}}{E} = \frac{\vec{u}}{c^2} \tag{18}$$

or

$$\vec{\beta} = \frac{\vec{u}}{c} = \frac{\vec{p}c}{E} \tag{19}$$

which give the dimensionless velocity $\vec{\beta} = \vec{u}/c$ in terms of $\vec{p}$ and $E$. Notice that $\vec{p}c$ has dimensions of energy, so the ratio $\vec{p}c/E$ is dimensionless, just like $\vec{\beta}$. Eq. (19) is one useful relation.

The other is

$$E^2 = (pc)^2 + \left(mc^2\right)^2 \tag{20}$$

This has the same form as the Pythagorean relation for a right triangle with sides $pc$ and $mc^2$ and hypotenuse $E$, though there isn't any deep meaning to this. To show that this is correct, we can plug in $E = \gamma mc^2$ and $p = \gamma mv$, solve for $\gamma^2$ and show that this reduces to the definition $\gamma^2 = (1 - v^2/c^2)^{-1}$.

*Units*

Since we are dealing with small things, we should use appropriate units. For example, if I ask how tall you are, you would tell me so many feet and so many inches. You could give me your height in miles or light years, but those are not appropriate units. Similarly, joules is not an appropriate unit for atomic and subatomic particles like electrons. The commonly used unit is the **electron volt** or eV. This is defined as the amount of work needed to move an electron ($q = -e = -1.60 \times 10^{-19}$ Coulomb) through a voltage drop of 1 volt ($\Delta V = -1$ volt); thus

$$1 \text{ eV} = q\Delta V = \left(-1.60 \times 10^{-19} \text{ C}\right) \times (-1 \text{ Volt}) = 1.60 \times 10^{-19} \text{ J} \tag{21}$$

Typical atomic energies are on the order of 1 eV or so, while those in nuclear physics are on the order of $10^6$ eV = 1 MeV.

Masses of particles are often given in eV with the understanding that they are referring to the rest energy $mc^2$. Techically, the mass is properly in units of eV$/c^2$. So, for example, the "mass" of the electron is given as 0.511 MeV, meaning $mc^2 = 0.511$ MeV. Similarly, when momentum $p$ is given in units of MeV, they really mean the quantity $pc$, or the units of $[p]$ is eV$/c$.

If you look up a table for the mass of atoms, the mass is often given in atomic mass units (denoted u). The conversion is

$$1 \text{ u} = 1.66 \times 10^{-27} \text{ kg} \tag{22}$$

**Conversion of Mass to Energy**

The famous equation $E = mc^2$ implies that if you can convert mass into energy, then you would get a lot of energy because the speed of light squared is so big. So does matter get converted to energy? The answer is yes. An example where this happens for atomic nuclei is when atomic nuclei break apart, either because they are unstable (radioactive) or because they are forced apart by an incoming particle (e.g., a neutron) in a process called fission. This is the principle behind atomic power plants like San Onofre.

Let's suppose that a nucleus at rest has a mass $M$, so that the initial energy is $Mc^2$. Then suppose the nucleus breaks apart due to radioactive decay in which pieces of the nucleus go flying apart. The energy then is a combination of the kinetic energy $K$ of the pieces and their rest energy. Suppose that the nucleus breaks into 2 pieces. Since total energy (including the rest mass energy) is conserved, we have

$$\begin{aligned} Mc^2 &= E_1 + E_2 \\ &= (K_1 + m_1c^2) + (K_2 + m_2c^2) \\ &= (K_1 + K_2) + (m_1c^2 + m_2c^2) \end{aligned} \tag{23}$$

The sum of the masses of the pieces $(m_1 + m_2)$ is less than the initial mass $M$, so there is a difference between the initial mass and the total final mass:

$$\Delta M c^2 = M c^2 - (m_1 + m_2) c^2 \qquad (24)$$

The missing mass is converted into the kinetic energy of the fragments flying apart:

$$\Delta M c^2 = K_1 + K_2 \qquad (25)$$

For example, your book considers the radioactive decay of $^{232}Th$ via the reaction

$$^{232}Th \rightarrow ^{228} Ra + ^{4} He \qquad (26)$$

The sum of the masses of $^{228}Ra$ and $^{4}He$ nuclei is 0.004 u smaller than the original $^{232}Th$ nucleus. Since 1 u $= 1.66 \times 10^{-27}$ kg, this corresponds to $7 \times 10^{-30}$ kg. This missing mass is converted into kinetic energy because the products ($^{228}Ra$ and $^{4}He$) fly apart. The amount of kinetic energy is

$$\begin{aligned} \Delta M c^2 &= \left(7 \times 10^{-30}\ \text{kg}\right)\left(3 \times 10^{8}\ \text{m/s}\right)^2 \\ &= 4\ \text{MeV} \end{aligned} \qquad (27)$$

If we reversed the process and forced the 2 pieces together to form the original nucleus, it would take work. The amount of energy we would have to expend would be $\Delta M c^2$. In other words, energy we put in would be converted into the additional mass so that the rest mass $M$ of the final nucleus would be greater than the sum $m_1 + m_2$.

So sometimes we need to add energy to merge or fuse 2 pieces together. However, sometimes, the opposite can happen. In particular, energy can be released when 2 things are brought together to form a more stable (lower energy) thing. For example, if we bring together an electron and a proton to form a hydrogen atom, then 13.6 eV is released. This is called the **binding energy** since is the amount of energy that would be required to pull apart the proton and electron in a hydrogen atom. If we add the mass $m_e$ of a free electron (free means it is floating around, and is not bound inside an atom), and the mass $m_p$ a free proton, the sum is 13.6 eV/$c^2$ greater than the mass $M_H$ of a hydrogen atom. This is another example of mass being converted into energy. In symbols:

$$m_e c^2 + m_p c^2 = M_H c^2 + B \tag{28}$$

where $B$ is the binding energy. So the binding energy corresponds to the rest energy of the missing mass:

$$B = \left(m_e c^2 + m_p c^2\right) - M c^2 \tag{29}$$

Similarly, when atoms are combined to form stable molecules, binding energy is released. Energy is also released when nuclei are forced together to form bigger stable nuclei in a process called fusion. For example, 4 hydrogen atoms can be fused to make 1 helium atom. This releases binding energy. This is what powers the sun and hydrogen bombs.

**Example: $\Lambda$ Decay**

The $\Lambda$ particle is a subatomic particle that can spontaneously decay into a proton and a negatively charged pion.

$$\Lambda \rightarrow p + \pi^- \tag{30}$$

The sum of the mass of the proton and the pion will be less than that of the $\Lambda$. In a certain experiment, the outgoing particles were traveling in the $+x$ direction with momenta $p_p =$ 581 MeV$/c$ and $p_\pi$=256 MeV$/c$. The rest masses are $m_p$ = 938 Mev$/c^2$ and $m_\pi$ = 140 MeV$/c^2$. Find the rest mass $m_\Lambda$ of the $\Lambda$.

To solve this problem, let us start with

$$E^2 = (pc)^2 + (mc^2)^2 \tag{31}$$

Solving for the mass of the $\Lambda$ gives

$$m_\Lambda = \frac{\sqrt{E_\Lambda^2 - (p_\Lambda c)^2}}{c^2} \tag{32}$$

So we need the energy and the momentum of the $\Lambda$ to find the mass. First we find the energy using Eq. (31)

$$E^2 = (pc)^2 + (mc^2)^2 \tag{33}$$

This gives $E_p$ = 1103 MeV and $E_\pi$ = 292 MeV. So

$$E_\Lambda = E_p + E_\pi = 1395 \text{ MeV} \tag{34}$$

Conservation of momentum gives

$$\mathbf{p}_\Lambda = \mathbf{p}_p + \mathbf{p}_\pi = 837 \text{ MeV}/c \tag{35}$$

where the momenta are all pointing along the positive $x$ axis. Now that we know $E_\Lambda$ and $p_\Lambda$, we can Eq. (32) to find the mass:

$$m_\Lambda = \frac{\sqrt{E_\Lambda^2 - (p_\Lambda c)^2}}{c^2} = 1116 \text{ MeV}/c^2 \tag{36}$$

This is how many masses of unstable subatomic particles are measured.

**Force in Relativity**

There are 2 definitions of force in classical mechanics:

$$\mathbf{F} = m\mathbf{a} \tag{37}$$

and

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{38}$$

These are equivalent because $\mathbf{p} = m\mathbf{u}$, so $d\mathbf{p}/dt = m\mathbf{a}$. But in relativity, $\mathbf{p} = \gamma m\mathbf{u}$, so $d\mathbf{p}/dt \neq m\mathbf{a}$. In relativity, the most convenient definition of force turns out to be

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{39}$$

Example 2.9 shows that this definition of force allows the work-energy theorem to remain valid. Namely,

$$\mathbf{F} \cdot d\mathbf{r} = dE \tag{40}$$

This says that if a mass $m$, acted on by a force $\mathbf{F}$, moves a small distance $d\mathbf{r}$, the change in its energy, $dE$, equals the work done by $\mathbf{F}$.

This definition of force allows us to keep the Lorentz force law intact:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{41}$$

One easy application of this equation is when a particle with charge $q$ moves with velocity $\mathbf{u}$ perpendicular to a uniform field $\mathbf{B}$. The particle's energy is constant because no work is done on it. To see this, look at the work-energy theorem. It turns out that the force is perpendicular to the direction of motion of the particle because the particle moves in a circle and the force points radially in towards the center of the circle. Therefore, since the energy $(E = \gamma mc^2)$ is constant, the velocity is constant in magnitude and only changes direction. To see that it moves in a circle, start with

$$\frac{d\mathbf{p}}{dt} = q\mathbf{u} \times \mathbf{B} \tag{42}$$

which we can write as

$$\gamma m \frac{d\mathbf{u}}{dt} = q\mathbf{u} \times \mathbf{B}. \tag{43}$$

Use acceleration $\mathbf{a} = d\mathbf{u}/dt$ and the fact that $\mathbf{u}$ is perpendicular to $\mathbf{B}$ to write

$$\gamma ma = quB \tag{44}$$

For motion in a circle of radius $R$, the centripetal acceleration $a = u^2/R$. The reason is the same in relativity as in classical mechanics. Using $\gamma mu = p$, we can solve for $R$ and write

$$R = \frac{p}{qB} \tag{45}$$

or

$$p = qBR \tag{46}$$

This provides a convenient way to measure the momentum of a particle of known charge $q$. We know the applied field $B$ and we can measure the radius $R$ of the circle.

For example, suppose we want to find the momentum of a proton moving perpendicular to a uniform magnetic field $B = 1.0$ tesla (T) in a circle of radius 1.4 m. We plug this into Eq. (46) using $q = e = 1.6 \times 10^{-19}$ C. This gives

$$\begin{aligned} p &= qBR = (1.6 \times 10^{-19} \text{ C}) \times (1.0 \text{ T}) \times (1.4 \text{ m}) \\ &= 2.24 \times 10^{-19} \text{ kg} - \text{m/s} \end{aligned} \tag{47}$$

Using the conversion 1 MeV/$c$=$5.34 \times 10^{-22}$ kg-m/s, we can change units:

$$p = (2.24 \times 10^{-19} \text{ kg} - \text{m/s}) \times \frac{1 \text{ MeV}/c}{5.34 \times 10^{-22} \text{ kg} - \text{m/s}} = 420\ MeV/c. \tag{48}$$

Since the proton's rest mass is known to be $m = 938$ MeV/$c^2$, we can find its energy from the "Pythagorean relation" to be

$$E = \sqrt{(pc)^2 + (mc^2)^2} = 1030 \text{ MeV} \tag{49}$$

**Massless Particles**

Is it possible to have massless particles? Classically, the answer is "no." Momentum ($\mathbf{p} = m\mathbf{u}$) and kinetic energy ($KE = mu^2/2$) are proportional to the mass $m$. If $m = 0$, then $p = E = 0$. One might think the same would be true relativistically since $\mathbf{p} = \gamma m\mathbf{u}$ and $E = \gamma mc^2$. Again, $p$ and $E$ are proportional to $m$ and $m = 0$. But what if the massless particle travels at the speed of light? Then $\gamma = \infty$, and $\gamma m = 0 \cdot \infty$ which is ill-defined.

It's better to go back to

$$E^2 = (pc)^2 + (mc^2)^2 \tag{50}$$

and

$$\beta = \frac{u}{c} = \frac{pc}{E} \tag{51}$$

If $m = 0$, then these equations reduce to

$$E = pc \tag{52}$$

and

$$\beta = \frac{pc}{E} = 1 = \frac{u}{c} \qquad \text{or} \qquad u = c \tag{53}$$

So the massless particle travels at the speed of light. Conversely, if we discovered a particle that traveled at speed $c$, then $\beta = u/c$ would equal $1{=}pc/E$. That would mean $E = pc$ and hence, that $m = 0$, i.e., it would be a massless particle. So massless particles travel at the speed of light, and particles that travel at $c$ must be massless. Particles with nonzero mass must travel at less than the speed of light.

Do massless particles exist? Yes. The best known massless particle is the **photon** which is a particle of light. As we will discuss more later, light can be described in 2 ways: (1) as an electromagnetic wave, and (2) as particles or packets of energy called photons.

To find the energy and momentum of a massless particle, we look at a reaction that it is involved in, e.g., photoionization of hydrogen in which a photon hits the electron in a hydrogen atom, causing the electron to be ejected:

$$\gamma + H \rightarrow e + p \tag{54}$$

where $\gamma$ is the symbol for a photon. We measure the momentum and energy of the particles with mass, then use conservation of energy to deduce the energy and momentum of the massless particle.

Another example of a massless particle is the graviton but there is no experimental evidence for the graviton. It used be accepted that the neutrino had no mass, but now there is experimental evidence that it does have a little bit of mass.

Example: Electron-Positron Annihilation

You've probably heard that when matter and antimatter collide, they annihilate and release a lot of energy. The **positron** is an anti-electron, i.e., it's the antimatter version of an electron. It is exactly like an electron except that it has a positive charge ($+e$). The positron has the same mass as an electron. Suppose a positron and an electron are both at rest with no momentum. So they only have their rest mass energies: $m_e c^2$ and $m_p c^2 = m_e c^2$. Then when they annihilate, 2 photons are produced. Conservation of momentum requires that their total momentum is zero, and the sum of their total energy is $2m_e c^2$:

$$\mathbf{p}_1 + \mathbf{p}_2 = 0 \tag{55}$$

and

$$E_1 + E_2 = 2m_e c^2 \tag{56}$$

So the photons that are created must have equal and opposite momenta:

$$\mathbf{p}_1 = -\mathbf{p}_2 \tag{57}$$

Since $E = pc$ for photons, the photons must have equal energies: $E_1 = E_2$, and

$$E_1 = E_2 = mc^2 = 0.511 \text{ MeV} \tag{58}$$

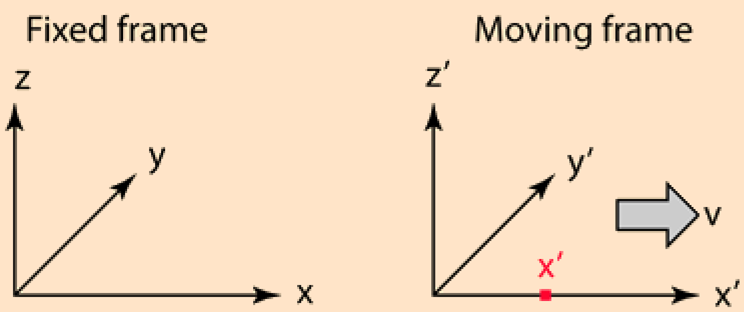
So all the rest mass energy is converted into electromagnetic energy.

Positrons are used in PET scans. PET stands for Positron Emission Tomography. This is a medical diagnostic probe in which a patient is injected with a radioactive positron emitter (like carbon 11) in a suitably chosen solution like glucose (sugar) that will tend to collect in the brain, for example. Once in the brain, the emitted positrons annihilate with nearby electrons, and photons are emitted that are detected by a ring of detectors. In this way, it is possible to create an accurate map of the areas of interest.

**When is nonrelativistic mechanics good enough?**

The relativistic equations all reduce to their classical counterparts in the limit of low speeds ($v \ll c$). When should one use the relativistic formulae and when is it good enough to use the classical equations? There is no clear-cut answer. It depends on what you need it for. For example, GPS (Global Positioning System) needs an accuracy of 1 ns, so relativistic effects must be taken into account. Table 2.4 of your book compares the values of the kinetic energy calculated with relativistic and classical formulas for various values of the velocity. As a general rule of thumb, you can ignore relativisitic effect if the speeds are $0.1c$ or less, or if a particle's kinetic energy $K$ is much less than 1% of its rest energy. (Recall that $K$ goes as $u^2$, so this is why a speed of $0.1c$ corresponds to 0.01% of the energy.) For speeds greater than $0.1c$ or kinetic energies greater than $0.01mc^2$, relativistic effects probably need to be taken into account, and relativistic equations should probably be used.

# Lorentz Transformation




$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The primed frame moves with velocity v in the x direction with respect to the fixed reference frame. The reference frames coincide at t=t'=0. The point x' is moving with the primed frame.

The reverse transformation is:

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

**Much of the literature of relativity uses the symbols β and γ as defined here to simplify the writing of relativistic relationships.**

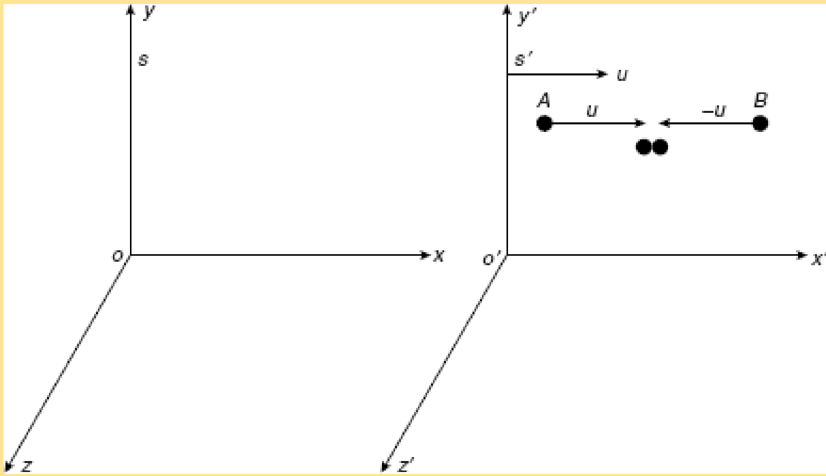
## Variation of Mass with Velocity

Consider two frames of references $S$ and $S'$. Further, $S'$ is moving with constant velocity $v$ along $X$-d consider the collision of two exactly similar balls A and B, each of mass $m$, moving in opposite dire collision they coalesce into one body.

Applying the, law of conservation of momentum on the collision of the balls in frame $s'$, we have

$$\begin{Bmatrix}\text{momentum of balls} \\ \text{before collision}\end{Bmatrix} = \begin{Bmatrix}\text{momentum of balls} \\ \text{after collision}\end{Bmatrix}$$

$$mu' + m(-u') = 0$$




After collision, the coalesced mass must be at rest in frame $S'$. Hence, it moves with velocity $v$ in frame $S$. Let $u_1, u_2$ be the velocities and $m_1$, $m_2$ be the masses of balls A and B, respectively, in frame $S$. Using the law of addition of velocities, the above velocities can be written as

$$u_1 = \frac{u' + v}{1 + \frac{u'v}{c^2}} \quad \text{......(1)} \qquad u_2 = \frac{-u' + v}{1 - \frac{u'v}{c^2}} \quad \text{.....(2)}$$

Applying the law of conservation of momentum on the collision of the balls in frame $s$, we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v \text{....(3)}$$

Substituting $u_1$ and $u_2$ values from Equations (1) and (2), we have

$$m_1\left[\frac{u'+v}{1+\frac{u'v}{c^2}}\right]+m_2\left[\frac{-u'+v}{1-\frac{u'v}{c^2}}\right]=[m_1+m_2]v \quad \text{or}$$

$$m_1\left[\frac{u'+v}{1+\frac{u'v}{c^2}}\right]-m_1v=m_2v-m_2\left[\frac{-u'+v}{1-\frac{u'v}{c^2}}\right] \quad \text{or}$$

$$m_1\left[\frac{u'+v}{1+\frac{u'v}{c^2}}-v\right]=m_2\left[v-\left(\frac{-u'+v}{1-\frac{u'v}{c^2}}\right)\right] \quad \text{or}$$

$$m_1\left[\frac{u'+v-v-\frac{u'v^2}{c^2}}{1+\frac{u'v}{c^2}}\right]=m_2\left[\frac{v-\frac{u'v^2}{c^2}+u'-v}{1-\frac{u'v}{c^2}}\right] \quad \text{or}$$

$$m_1\left[\frac{u'-\frac{u'v^2}{c^2}}{1+\frac{u'v}{c^2}}\right]=m_2\left[\frac{u'-\frac{u'v^2}{c^2}}{1-\frac{u'v}{c^2}}\right] \quad \text{or}$$

$$\frac{m_1}{m_2}=\frac{1+\frac{u'v}{c^2}}{1-\frac{u'v}{c^2}} \quad \text{.........(4)}$$

The above equation makes a relationship between the masses of balls in frame *S* and their velocities in frame *S*'. Now, to obtain relation between masses of balls and their velocities in frame *S*, we proceed as follows. Squaring Equation (1)

$$u_1^2=\frac{(u'+v)^2}{\left[1+\frac{u'v}{c^2}\right]^2}$$

and using the above equation, the value

$1-\frac{u_1^2}{c^2}$ is

$$1-\frac{u_1^2}{c^2}=1-\frac{(u'+v)^2}{c^2\left[1+\frac{u'v}{c^2}\right]^2}=1-\frac{\frac{(u'+v)^2}{c^2}}{\left[1+\frac{u'v}{c^2}\right]^2}$$

$$=\frac{\left[1+\frac{u'v}{c^2}\right]^2-\frac{(u'+v)^2}{c^2}}{\left[1+\frac{u'v}{c^2}\right]^2}$$

$$=\frac{1+\frac{u'^2v^2}{c^4}+\frac{2u'v}{c^2}-\frac{u'^2}{c^2}-\frac{v^2}{c^2}-\frac{2u'v}{c^2}}{\left[1+\frac{u'v}{c^2}\right]^2}$$

$$=\frac{1-\frac{v^2}{c^2}-\frac{u'^2}{c^2}\left[1-\frac{v^2}{c^2}\right]}{\left[1+\frac{u'v}{c^2}\right]^2}$$

$$=\frac{\left[1-\frac{v^2}{c^2}\right]\left[1-\frac{u'^2}{c^2}\right]}{\left[1+\frac{u'v}{c^2}\right]^2}$$

Therefore,

$$1-\frac{{u_1}^2}{c^2}=\frac{(1-\frac{u'^2}{c^2})(1-\frac{v^2}{c^2})}{(1+\frac{u'v}{c^2})^2} \quad ...(5)$$

Similarly, using equation (2) we get

$$1-\frac{{u_2}^2}{c^2}=\frac{(1-\frac{u'^2}{c^2})(1-\frac{v^2}{c^2})}{(1-\frac{u'v}{c^2})^2} \quad ...(6)$$

Dividing Equation (6) by Equation (5) and taking square root throughout, we have

$$\frac{\sqrt{1-\frac{{u_2}^2}{c^2}}}{\sqrt{1-\frac{{u_1}^2}{c^2}}}=\frac{1+\frac{u'v}{c^2}}{1-\frac{u'v}{c^2}} \quad ...(7)$$

Comparing Equations (4) and (7), we have

$$\frac{\sqrt{1-\frac{u_2{}^2}{c^2}}}{\sqrt{1-\frac{u_1{}^2}{c^2}}}=\frac{m_1}{m_2} \quad ....(8)$$

Suppose, $m_2$ is at rest in frame $s$, then $u_2 = 0$ and $m_2 = m_0$ (say) where $m0$ is the rest mass of the ball B, thenEquation (19.48) becomes

$$\frac{1}{\sqrt{1-\frac{u_1{}^2}{c^2}}}=\frac{m_1}{m_o} \quad ....(9)$$

As both the balls are similar, hence the rest masses of both balls are the same, so we can write the rest mass of $m_2$ is equal to rest mass of $m_1$, that is equal to $m_o$. Then, Equation (9) becomes

$$\frac{m_o}{\sqrt{1-\frac{u_1{}^2}{c^2}}}=m_1 \quad ....(10)$$

Here, $m_1$ is the mass of ball A when it is moving with velocity $u_1$ in frame $s$. After collision, the coalescent mass containing mass of ball A moves with velocity $v$ in frame $s$.

In general, if we take the mass of ball A as $m$, when it is moving with velocity $v$ in frame $s$, then

$$\frac{m_o}{\sqrt{1-\frac{u^2}{c^2}}}=m \quad .......(11)$$

where $m_0$ is the rest mass of the body and $m$ is the effective mass.

Equation (11) is the relativistic formula for the variation of mass with velocity. Here, we see some special cases:

**Case (i):** When the velocity of the body, $v$ is very small compared to velocity of height, $c$, then $v^2/c^2$ is negligible compared to one. Therefore,

$m = m_{o}*$

**Case (ii):** If the velocity of the body $v$ is comparable to the velocity of light $c$,

then $\sqrt{1-v^2/c^2}$ i is less than one, so, $m > m_o$.

The mass of a moving body appears greater than its rest mass.

**Case (iii):** Suppose the velocity of a body is equal to velocity of light, $c$, then, it possess infinite mass.

The effective mass of particles has been experimentally verified by using particle accelerators in case of electrons and protons by increasing their velocities very close to velocity of light.

Aryabhatta was an extraordinary teacher and scholar who had immense knowledge about mathematics and astronomy. He suggested the heliocentric theory which proved that the sun is located in the centre of the solar system and all the planets revolve around it. In fact he made this discovery way before Copernicus made this discovery in the West.

Aryabhatta was born in Kerala and lived from 476 AD to 550 AD, he completed his education from the ancient university of Nalanda and later he moved to Bihar and continued his studies in the great centre of learning located in close proximity to Kusumapura in Bihar and lived in Taregana District in Bihar in the late 5th and early 6th century.

### His contribution to the astronomy

The astronomical calculations and deductions suggested by Aryabhatta are extraordinary by the fact that he didn't have any modern equipment or instrument to do it. He had a very sharp brain and his dedication and hard work led him to solve the various mysteries of the solar system. He also deduced that the earth is round in shape and rotates along its own axis, which forms the existence of day and night. Many superstitious beliefs were challenged by him and he presented scientific reasons to prove them wrong.

He also said that the moon has no light and shines because it reflects light from the sun. He also proved wrong the false belief that eclipse is caused because of the shadows formed by the shadows cast by the earth and the moon. Aryabhatta used epicycles in a similar manner to the Greek Philosopher Ptolemy to illustrate the inconsistent movement of some planets. This great astronomer wrote the famous treatise Aryabhatiya, which was based on astronomy in 499 AD. This treatise was acknowledged as a masterpiece. In honour of this excellent work Aryabhatta was made head of the Nalanda University by the Gupta ruler Buddhagupta.

# Gilbert and the amber force: 1600

The year 1600 is a good one for William Gilbert. He is appointed court physician to Queen Elizabeth, and the summary of his life-long research into magnetism is published as *De magnete, magneticisque corporibus, et de magno magnete tellure* (Of the magnet, of magnetic bodies, and of the earth as a great magnet).

As the title states, Gilbert's work has led him to the grand conclusion that compasses behave as they do because the earth itself is a vast magnet. He introduces the term 'magnetic pole', and states that the magnetic poles lie near the geographic poles.

Gilbert describes useful practical experiments, revealing how iron can be magnetized for use in compasses without relying on rare and expensive lodestone. Hammering the metal will do the trick, if the iron is correctly aligned with the earth's magnetic field.

Gilbert's researches also involve him in the mysterious property of amber, recognized 2000 years previously by Greek scientists. He identifies this as a force and coins a term for it from *elektron*, the Greek for amber. He calls it, in an invented Latin phrase, *vis electrica*- the 'amber force'. Electricityhas found its name.

# Galileo and the Discorsi: 1634-1638

In December 1633 Galileo is place under house arrest, on the pope's orders, because of his work on astronomy. Finding himself confined to his small estate at Arcetri near Florence, his response is typically positive. He settles down to explain and prove his early and less controversial discoveries in the mechanical sciences.

Two are particularly well known. The first he is said to have observed as a student in Pisa, when he watches a lamp swinging in the cathedral, times it by his own pulse, and discovers that each swing takes the same amount of time regardless of how far the lamp travels. At Arcetri he demonstrates this principle of the pendulum experimentally, and suggests its possible use in relation to clocks.

His other most famous discovery in physics, proved theoretically in about 1604 when he is professor of mathematics in Padua, is that bodies falling in a vacuum do so at the same speed and at a uniform rate of acceleration. (There is as yet no vacuum in which to demonstrate this law, but Boyle is able to do so later in the century.) While at Padua Galileo also works out the laws of ballistics, or the dynamics of objects moving through the air in a curve rather than falling directly to earth.

Written up and proved mathematically during 1634, these theorems are published in Leiden in 1638 as the *Discorsi e dimostrazioni matematichè intorno à due nuove scienze attenenti alla mecanica et i movementi locali*.

Galileo's title claims to introduce two new sciences, mechanics and 'local movements', and his book stands at the start of mathematical physics. He is the first to use mathematics to understand and explain physical phenomena, and he is the first to make rigorous use of experiment to check results provided by theory. The attractive notion of his dropping weights from the leaning tower of Pisa, to check on the behaviour of falling bodies, is only a legend. But he certainly, if more mundanely, rolls balls down inclined planes for the same purpose.

Galileo provides the foundation on which Newton (born in the year of Galileo's death) soon builds.

# Barometer and atmospheric pressure: 1643-1646

Like many significant discoveries, the principle of the barometer is observed by accident. Evangelista Torricelli, assistant to Galileo at the end of his life, is interested in why it is more difficult to pump water from a well in which the water lies far below ground level. He suspects that the reason may be the weight of the extra column of air above the water, and he devises a way of testing this theory.

He fills a glass tube with mercury. Submerging it in a bath of mercury, and raising the sealed end to a vertical position, he finds that the mercury slips a little way down the tube. He reasons that the weight of air on the mercury in the bath is supporting the weight of the column of mercury in the tube.

If this is true, then the space in the glass tube above the mercury column must be a vacuum. This plunges him into instant controversy with traditionalists, wedded to the ancient theory - going as far back as Aristotle - that 'nature abhors a vacuum'. But it also encourages von Guericke, in the next decade, to develop the vacuum pump.

The concept of variable atmospheric pressure occurs to Torricelli when he notices, in 1643, that the height of his column of mercury sometimes varies slightly from its normal level, which is 760 mm above the mercury level in the bath. Observation suggests that these variations relate closely to changes in the weather. The barometer is born.

With the concept thus established that air has weight, Torricelli is able to predict that there must be less atmospheric pressure at higher altitudes. It is not hard to imagine an experiment which would test this, but the fame for proving the point in 1646 attaches to Blaise Pascal - though it is not even he who carries out the research.

Having a weak constitution, Pascal persuades his more robust brother-in-law to carry a barometer to different levels of the 4000-foot Puy de Dôme, near Clermont, and to take readings. The brother-in-law descends from the mountain with the welcome news that the readings were indeed different. Atmospheric pressure varies with altitude.

# Von Guericke and the vacuum: 1654-1657

Spectators in the town square of Regensburg, on 8 May 1654, are treated to perhaps the most dramatic demonstration in the history of science. Otto von Guericke, burgomaster of Magdeburg and part-time experimenter in physics, is about to demonstrate the reality of a vacuum.

Aristotle declared that there can be no such thing as empty space, but von Guericke has spent several years perfecting an air pump which can achieve just that. He now produces two hollow metal hemispheres and places them loosely together. There is no locking device. Von Guericke works for a while at his pump, attached by a tube to one of the hemispheres. He then signals that he is ready.

Sixteen horses are harnessed in two teams of eight. Each team is attached to one of the hemispheres. Whipped in opposite directions, the horses fail to pull the sphere apart. Yet when von Guericke undoes a nozzle of some kind, the two halves separate easily.

A mysterious point has been very forcefully made. Von Guericke's experiments are first described in a book of 1657 (*Mechanica Hydraulica-Pneumatica* by Kaspar Schott). The vacuum thus becomes available to the scientific community as an experimental medium. Von Guericke himself uses it to demonstrate that a bell is muffled in a vacuum and a flame extinguished. Robert Boyle, too, soon borrows the device.

# Robert Boyle: 1661-1666

The experimental methods of modern science are considerably advanced by the work of Robert Boyle during the 1660s. He is skilful at devising experiments to test theories, though an early success is merely a matter of using von Guericke's air pump to create a vacuum in which he can observe the behaviour of falling bodies.

He is able to demonstrate the truth of Galileo's proposition that all objects will fall at the same speed in a vacuum.

But Boyle also uses the air pump to make significant discoveries of his own - most notably that reduction in pressure reduces the boiling temperature of a liquid (water boils at 100° at normal air pressure, but at only 46°C if the pressure is reduced to one tenth).

Boyle's best-known experiment involves a U-shaped glass tube open at one end. Air is trapped in the closed end by a column of mercury. Boyle can show that if the weight of mercury is doubled, the volume of air is halved. The conclusion is the principle known still in Britain and the USA as Boyle's Law - that pressure and volume are inversely proportional for a fixed mass of gas at a constant temperature.

Boyle's most famous work has a title perfectly expressing a correct scientific attitude. *The Sceptical Chymist*appears in 1661. Boyle is properly sceptical about contemporary theories on the nature of matter, which still derive mainly from the Greek theory of four elements.

His own notions are much closer to the truth. Indeed it is he who introduces the concept of the element in its modern sense, suggesting that such entities are 'primitive and simple, or perfectly unmingled bodies'. Elements, as he imagines them, are 'corpuscles' of different sorts and sizes which arrange themselves into compounds - the chemical substances familiar to our senses. Compounds, he argues, can be broken down into their constituent elements. Boyle's ideas in this field are further developed in his *Origin of Forms and Qualities*(1666).

Chemistry is Boyle's prime interest, but he also makes intelligent contributions in the field of pure physics.

In an important work of 1663, *Experiments and Considerations Touching Colours*, Boyle argues that colours have no intrinsic identity but are modifications in light reflected from different surfaces. (This is demonstrated within a few years by Newton in his work on the spectrum.)

As a man of his time, Boyle is as much interested in theology as science. It comes as a shock to read his requirements for the annual Boyle lecture which he founds in his will. Instead of discussing science, the lecturers are to prove the truth of Christianity against 'notorious infidels, viz., atheists, theists, pagans, Jews and Mahommedans'. The rules specifically forbid any mention of disagreement among Christian sects.

# Newton in the garden: 1665-1666

The Great Plague of 1665 has one unexpected beneficial effect. It causes Cambridge university to close as a precaution, sending the students home. A not particularly distinguished member of Trinity College, who has recently failed an examination owing to his feeble geometry, travels home to the isolated Woolsthorpe Manor in Lincolnshire.

He spends there the greater part of eighteen months, one of the most productive periods in scientific history. With time for uninterrupted concentration, he works out the binomial theorem, differential and integral calculus, the relationship between light and colour and the concept of gravity. The student is the 22-year-old Isaac Newton.

The famous detail of the falling apple in the garden of Woolsthorpe Manor, as the moment of truth in relation to gravity, provides the perfect seed for a popular legend. But the story is first told in the next century, by Voltaire, who claims to have had it from Newton's step-niece. In reality it is the moon which prompts Newton's researches into gravity.

Meanwhile his discoveries in relation to light and colour bring him his first fame.

# Newton and Opticks: 1666-1672

Returning to Cambridge in 1666, and discussing there his new discoveries, Newton wins an immediate reputation. In 1669, when still short of his twenty-seventh birthday, he is elected the Lucasian professor of mathematics. His lectures and researches are mainly at this stage to do with optics. He invents for his purposes a new and more powerful form of telescope using mirrors (the reflecting telescope, which becomes the principle of all the most powerful instruments until the introduction of radio astronomy).

In 1672 he presents a telescope of this kind to the Royal Societyand is elected a member. Later in this same year he describes for the Society his experiments with the prism.

In this famous piece of research Newton directs a shaft of sunlight through a prism. He finds that it spreads out and splits into separate colours covering the full range of the spectrum. If he directs these coloured rays through a reverse prism, the light emerging is once again white. However if he isolates any single colour, by sending it to the second prism through a narrow slot, it will emerge as that same colour, unchanged.

It has often previously been observed that light passing through a medium such as a bowl of water can change colour, but it has been assumed that this colour is imparted by the glass or water.

Newton's reversible experiment proves that the phenomenon is an aspect of light itself. Different wavelengths of light have different angles of refraction, with the result that the prism separates them. White light, containing all the wave lengths, can be transformed back and forth. Light of a single wave length and colour can only remain itself.

It follows from this that the perceived colour of different substances derives from the particular wavelengths of light which they reflect to the eye; or, in Newton's words, that 'natural bodies are variously qualified to reflect one sort of light in greater plenty than another'. The sciences of colour and of spectrum analysis begin with this work, which Newton eventually publishes in 1704 as *Opticks*.

# Newton and gravity: 1684-1687

In 1684 Edmund Halleyvisits Newton in Cambridge. Hearing his ideas on the motion of celestial bodies, he urges him to develop them as a book. The result is the *Principia Mathematica* (in full *Philosophiae Naturalis Principia Mathematica*, Mathematical Principles of Natural Philosophy), published in 1687. When lack of funds in the Royal Society seems likely to delay the project, Halley pays the entire cost of printing himself.

The book, one of the most influential in the history of science, derives from the young Newton's speculations about the moon during his time at Woolsthorpe Manor two decades earlier.

The question which stimulated his thoughts was this: what prevents the moon from flying out of its orbit round the earth, just as a ball being whirled on a string will fly away if the string breaks? The ball, in such an event, flies off at a tangent. Newton reasons that the moon can be seen as perpetually falling from such a tangent into its continuing orbit round the earth.

He calculates mathematically by how much, on such an analogy, the moon is falling every second. He then uses these figures to calculate, on the same principle, the probable speed of a body falling in the usual way in our own surroundings. He finds that theory and reality match, in his own words, 'pretty nearly'.

The word gravity is already in use at this time, to mean the quality of heaviness which causes an object to fall. Newton demonstrates its existence now as a universal law: 'Any two particles of matter attract one another with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them.'

With this observation he introduces the great unifying principle of classical physics, capable of explaining in one mathematical law the motion of the planets, the movement of the tides and the fall of an apple.

# The Leyden jar: 1745-1746

The researches of William Gilbert, at the start of the 17th century, lead eventually to simple machines with which enthusiasts can generate an electric charge by means of friction. The current generated will give a stimulating frisson to a lady's hand, or can be discharged as a spark.

In 1745 an amateur scientist, Ewald Georg von Kleist, dean of the cathedral in Kamien, makes an interesting discovery. After partly filling a glass jar with water, and pushing a metal rod through a cork stopper until it reaches the water, he attaches the end of the nail to his friction machine.

After a suitable amount of whirring, the friction machine is disconnected. When Kleist touches the top of the nail he can feel a slight shock, proving that static electricity has remained in the jar. It is the first time that electricity has been stored in this way, for future discharge, in the type of device known as a capacitor.

In 1746 the same principle is discovered by Pieter van Musschenbroek, a physicist in the university of Leyden. As a professional, he makes much use of the new device in laboratory experiments. Though sometimes called a Kleistian jar, it becomes more commonly known as the Leyden jar.

Within a year or two an improvement is made which gives the capacitor its lasting identity. The water in the vessel is replaced by a lining of metal foil, with which the metal rod projecting from the jar is in contact. Another layer of metal foil is wrapped round the outside of the jar. The two foils are charged with equal amounts of electricity, one charge being positive and the other negative.

The principle of plates bearing opposite charges, and separated only by a narrow layer of insulation, remains constant in the development of capacitors - much used in modern technology.

# Watson and Franklin: 1745-1752

In 1745 the Royal Society in London awards its highest honour, the Copley medal, to William Watson for his researches into electricity. It is the fashionable subject of the moment, and is about to become more so with the development of the Leyden jar.

In 1747 Watson sets up an ambitious experiment to discover the speed at which electricity travels. He arranges an electrical circuit more than two miles long, linking the positive and negative metal foils of a Leyden jar. There seems to be no measurable difference between the completion of the circuit and the moment when an observer at the middle of the loop feels the shock. Watson concludes that electricity is 'instantaneous'.

His conclusion is not an accurate description of the flow of electricity, but the experiment is nonetheless impressive. As the leading figure in electrical research, Watson is now in touch with an enthusiastic experimenter on the other side of the Atlantic, Benjamin Franklin.

Watson and Franklin independently arrive at a new and correct concept of electricity - that instead of being created by friction between two surfaces, it is something transferred from one to the other, electrically charging both. They see electricity as the flow of a substance which can be neither created nor destroyed. The total quantity of electricity in an insulated system remains constant.

Franklin, a scientist with a popular touch, coins several of the terms which are now standard - positive and negative, conductor, battery (in the sense of a series of Leyden jars linked for simultaneous charge or discharge). His papers on the subject, gathered and published in 1751 as *Experiments and Observations on Electricity*, become the first (and perhaps only) electrical best-seller. Widely read in successive English editions, and translated into French, German and Italian, this short book makes Franklin an international celebrity.

His reputation is further enhanced, in the following year, when he devises history's most dramatic, and dangerous, electrical experiment.

The new Leyden jars are powerful enough to generate a spark which is both visible and audible. It occurs to many that this effect may be the same as that generated in nature in the form of lightning. Franklin invents a way of testing this idea.

In Philadelphia, in 1752, he adds a metal tip to a kite and flies it on a wet string into a thunder cloud. The bottom of the string is attached to a Leyden jar. The point is made when the Leyden jar is successfully charged. For the popular audience Franklin makes the effect visible. He attracts sparks from a key attached to the line. His fame soars. (But the next two people attempting the experiment are killed.)

In conducting his experiment, Franklin already has in mind a practical application if the science proves correct. He reasons that if celestial electricity can be attracted to a metal point, then a rod projecting from the top of a church steeple, connected by a metal strip to the earth, could serve as a conductor for any stroke of lightning and thus save the building from harm.

When the British army proposes to construct a magazine at Purfleet for the storage of gunpowder, William Watson recommends that this highly explosive building be protected by one of Benjamin Franklin's lightning conductors. The proposal is accepted. The science of electricity finds the first of its myriad eventual roles in everyday life.

# Joseph Black and latent heat: 1761

Joseph Black notices that when ice melts it absorbs a certain amount of heat without any rise in temperature. He reasons that the heat must have combined with the particles of ice and still be present in the water at 0°C. Heat of this kind (as Cavendish later perceives) consists of greater activity among the molecules, in a form of energy which will be transferred again if the water freezes.

Black calls this phenomenon latent heat, and teaches it in his lectures at the university of Glasgow from 1761. An important discovery in itself, it also enables him to be the first to distinguish between heat (energy transferred from a warmer to a colder object) and temperature (the amount of energy present at a given moment).

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